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EDITED BY

T. A. A. BROADBENT, M.A.

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ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION, 1938.

THE Annual Meeting of the Mathematical Association in 1938 will be held on January 4th and 5th, at the Institute of Education, Southampton Row, London, W.C. 1.

LOGARITHMS BY INTERPOLATION.*

BY N. M. GIBBINS.

THE method of introducing logarithms given in the *Algebra Report* consists in taking the powers of 1·1, which are easily calculated and are so closely packed that common logarithms can be deduced by proportional parts correct to three decimal places. It forms an interesting set of lessons for post-certificate pupils to discuss why this is so, and to extend the method with a view to obtaining greater accuracy. Furthermore, one of the most important principles in pure mathematics is amply illustrated—that of sandwiching a function between two bounds. It is assumed that the expansion of $\log_e(1+x)$ is known by the pupils and also its differential coefficient.

The Report says that it is graphically obvious that all the logarithms obtained by proportional parts are too small; and it is also clear that the further the proportional part diverges from one half

* A paper read at the "Members' Topics" meeting of the London Branch on 25th January, 1936.

the more accurate the result. For example, if we take the base 2, $\log 1{\cdot}024 > {\cdot}024$ and $\log 1000 > 9\frac{488}{512}$. We thus obtain

$$\cdot 30072 < \log_{10} 2 < \cdot 30142,$$

so that $\log_{10} 2 = \cdot 301$ correct to three decimal places.

The first thing to do is to put this intuitive idea on an analytical basis. As regards notation, we take $1+b$ as the base, $(0 < b < 1)$; and then the proportional part of $\log_{1+b}(1+bx)$ is x, $(0 < x < 1)$. The differential coefficient of $x^{-1}\log_e(1+bx)$ is easily found to be negative. Hence, since x is less than 1,

$$x^{-1}\log_e(1+bx) > \log_e(1+b), \quad \text{or} \quad \log_{1+b}(1+bx) > x.$$

It is also easy to lead up to the function $(x^{-1}+\frac{1}{2}b)\log_e(1+bx)$, whose differential coefficient is positive. Hence the function is less than $(1+\frac{1}{2}b)\log_e(1+b)$, that is

$$\log_{1+b}(1+bx) < x(1+\tfrac{1}{2}b)/(1+\tfrac{1}{2}bx).$$

We also have

$$x(1+\tfrac{1}{2}b)/(1+\tfrac{1}{2}bx) - x = \tfrac{1}{2}bx(1-x)/(1+\tfrac{1}{2}bx) < \tfrac{1}{2}bx(1-x),$$

the maximum value of which is $\frac{1}{8}b$. If $b = \frac{1}{10}$, as in the Report, $\log_{1\cdot 1} 10 > 24$, from the table there given. Hence the maximum error in a common logarithm is less than $\frac{1}{24}$ of $\frac{1}{80}$, or $\frac{1}{1920}$, which is not sufficiently small for three-figure accuracy. But if $x = \cdot 4$ or $\cdot 6$ the error is less than $\frac{1}{24}$ of $(\cdot 4 \times \cdot 6)/20$ or $\frac{1}{2000}$. Examination of the proportional parts found in the Report shows that none of them lies between $\cdot 4$ and $\cdot 6$, and so the three-figure accuracy claimed is established.

The Report suggests that to use the base 1·01 to obtain better results would be very tedious if a table of powers of 1·01 had to be compiled: to reach log 10 we should have to perform 231 operations. It is clear then that low cunning must take the place of hard labour, and as an example of the method to be adopted we proceed to find common logarithms from those to the base $1\frac{1}{2}$ by help of the table $1\frac{1}{2}, 1$; $2\frac{1}{4}, 2$. By proportional parts $\log 2 = 1\frac{2}{3}$, $\log 2\frac{1}{2} = 2\frac{2}{9}$. Hence $\log 5 = 3\frac{8}{9}$, $\log 10 = 5\frac{5}{9}$. Thus

$$\log_{10} 2 = 1\tfrac{2}{3} \div 5\tfrac{5}{9} = \tfrac{3}{10}, \quad \text{and} \quad \log_{10} 5 = 3\tfrac{8}{9} \div 5\tfrac{5}{9} = \tfrac{7}{10}.$$

Also $\log 3 = 2\frac{2}{3}$, whence $\log_{10} 3 = 2\frac{2}{3} \div 5\frac{5}{9} = \cdot 48$; and so on.

Now putting $b = \frac{1}{2}$ in the upper bound $x(1+\frac{1}{2}b)/(1+\frac{1}{2}bx)$ we have that log 2 lies between $1\frac{2}{3}$ and $1\frac{5}{7}$, and $\log 2\frac{1}{2}$ lies between $2\frac{2}{9}$ and $2\frac{5}{19}$.

Hence $\quad \log 2 = 1\frac{2}{3} + \epsilon, \quad$ where $\quad 0 < \epsilon < \frac{1}{21}$;

and $\quad \log 2\frac{1}{2} = 2\frac{2}{9} + \epsilon', \quad$ where $\quad 0 < \epsilon' < \frac{7}{191}$.

It simplifies the arithmetic to replace $\frac{1}{21}$ and $\frac{7}{191}$ by the larger $\frac{1}{18}$, which otherwise is not in our favour. We have

$$\log 10 = 5\tfrac{5}{9} + 2\epsilon + \epsilon'.$$

Hence $\quad \log_{10} 2 = (1\frac{2}{3} + \epsilon)/(5\frac{5}{9} + 2\epsilon + \epsilon'),$

and for a given value of ϵ' this increases with ϵ since its differential coefficient with respect to ϵ is positive.

Hence $$\frac{1\frac{2}{3}}{5\frac{5}{9}+\epsilon'} < \log_{10} 2 < \frac{1\frac{2}{3}+\frac{1}{18}}{5\frac{5}{9}+\frac{1}{9}+\epsilon'};$$

and *a fortiori*, $$\frac{1\frac{2}{3}}{5\frac{5}{9}+\frac{1}{18}} < \log_{10} 2 < \frac{1\frac{2}{3}+\frac{1}{18}}{5\frac{5}{9}+\frac{1}{9}},$$

or $$\tfrac{30}{101} < \log_{10} 2 < \tfrac{31}{102},$$

that is, $$\tfrac{3}{10} - \tfrac{3}{1010} < \log_{10} 2 < \tfrac{3}{10} + \tfrac{4}{1020}.$$

Hence $\log_{10} 2 = \cdot 30$ for certain. Similarly $\log_{10} 3$ lies between $\cdot 4803$ and $\cdot 4765$, and so is $\cdot 48$ for certain.

To return to the base 1·01 we need the following table: 1·01, 1; 1·0201, 2; 1·030301, 3; 1·04060401, 4. To check up the errors we do not need the full upper bound $x(1+\frac{1}{2}b)/(1+\frac{1}{2}bx)$, but only its numerator, which is greater. Thus for any proportional part x, double the maximum error is $x/100$, which can be written down. It has further been planned that the errors should be cumulative.

We now take suitable multiples of 2, 3 and 10, namely 1·024; 81/80, or 1·0125; and 25/24, or 1·0416667: then calculate their logarithms to base 1·01 by proportional parts, and solve the resulting equations in log 2, log 3 and log 10, keeping count of the errors as we go on. For example, we have

$$\log 1\cdot 024 = \log (1\cdot 01)^2 \times \frac{1\cdot 024}{1\cdot 0201}$$

$$= 2 + \log\left(1 + \frac{\cdot 0039}{1\cdot 0201}\right) = 2 + \log\left(1 + \frac{\cdot 3823}{100}\right) \simeq 2\cdot 3823,$$

with double maximum error $\cdot 003823$. Proceeding in this way for the other two numbers, and putting $\log 2 = x$, $\log 3 = y$, $\log 10 = z$, we have the following calculations:

	Double error in excess
$\log 1\cdot 024 = 10x - 3z = 2 + \cdot 3823,$	$\cdot 003823.$
$\log 1\cdot 0125 = -3x + 4y - z = 1 + \cdot 2475,$	$\cdot 002475.$
$\log 25/24 = -5x - y + 2z = 4 + \cdot 1021,$	$\cdot 001021.$

Multiply these equations respectively by 23, 10, 40 and add.

$z = 46 + 8\cdot 793,$	$\cdot 08793.$
$+ 10 + 2\cdot 475,$	$\cdot 02475.$
$+ 160 + 4\cdot 084,$	$\cdot 04084.$
$= 216 + 15\cdot 352,$	$\cdot 15352.$

The error is less than $\cdot 077$.

Hence z lies between 231·35 and 231·43, and so $z = 231\cdot 4$.

Since $z > 200$ we have from the first equation that the error in x/z or $\log_{10} 2$ is less than $\frac{1}{10} \times \frac{1}{200} \times \frac{1}{2}(\cdot 004)$ or 1×10^{-6}.

Hence $\log_{10} 2 = \cdot 3 + \cdot 23823/231\cdot 4 = \cdot 30103$ to five places.

From the first two equations,

$$\begin{aligned} 40y-19z &= 6+1\cdot147 \\ &\quad +10+2\cdot475 \\ &= 16+3\cdot622, \end{aligned}$$

with error less than $\cdot02$. Hence the error in y/z or $\log_{10} 3$ is less than $\frac{1}{40}\times\frac{1}{200}\times\cdot02$, or $2\cdot5\times10^{-6}$.

Hence $\dfrac{y}{z}=\dfrac{19}{40}+\dfrac{19\cdot622}{231\cdot4\times40}=\cdot47712$, to five places.

In calculating other logarithms we have to be sure that the proportional part is so chosen that $x(1-x)/200\times231<5/10^6$, that is, that x does not lie between $\cdot36$ and $\cdot64$. Furthermore, as it is proposed to obtain the common logarithms of other primes from, *e.g.* 1·008, 100/99, 1·04, 1·02, 96/95, 24/23, and so on, it is necessary to obtain $\log_{10} 2$ and $\log_{10} 3$ to a higher degree of accuracy. To do this we need a more accurate lower bound than the proportional part. We can, indeed, obtain any number of bounds, alternately lower and upper, and successively increasing in accuracy by differentiating the function

$$x^{-1}(a_0+a_1bx+a_2b^2x^2+\ldots)\log_e(1+bx),$$

where we add in one term at a time. We have already found that $a_0=1$, $a_1=\frac{1}{2}$, and we shall find that $a_2=-\frac{1}{12}$, $a_3=\frac{1}{24}$, and so on. The new lower bound is therefore

$$\frac{x\left(1+\dfrac{b}{2}-\dfrac{b^2}{12}\right)}{1+\dfrac{bx}{2}-\dfrac{b^2x^2}{12}}=\frac{x\left\{\left(1+\dfrac{b}{2}\right)\left(1-\dfrac{b^2}{12}\right)+\dfrac{b^3}{24}\right\}}{1+\dfrac{bx}{2}-\dfrac{b^2x^2}{12}}>\frac{x(1+\frac{1}{2}b)}{1+\frac{1}{2}bx}\cdot\left(1-\frac{b^2}{12}\right).$$

Thus the errors, when we calculate logarithms to base 1·01 from the first upper bound, are always less than 1/120000 of the bound. When $b=\cdot01$, the upper bound is $201x/(200+x)$,

$$\log 1\cdot024=2+201\times\frac{\cdot39}{1\cdot0201}\bigg/\left(200+\frac{\cdot39}{1\cdot0201}\right)=2+\frac{78\cdot39}{204\cdot41},$$

$$\log 1\cdot0125=1+201\times\frac{\cdot25}{1\cdot01}\bigg/\left(200+\frac{\cdot25}{1\cdot01}\right)=1+\frac{50\cdot25}{202\cdot25},$$

$$\log 25/24=4+(201\times\cdot10212)/(200\cdot10212).$$

Hence
$$\begin{aligned} 10x\qquad\; -3z &= 2\cdot38350, \\ -3x+4y-\;\; z &= 1\cdot24845, \\ -5x-\;\; y+2z &= 4\cdot10258. \end{aligned}$$

Thus
$$\begin{aligned} z &= 46+\;\;8\cdot8205 \\ &\quad +\;10+\;\;2\cdot4845 \\ &\quad +160+\;\;4\cdot1032 \\ &= 216+15\cdot4082, \end{aligned}$$

with error in defect less than $15\cdot4082\div120{,}000=\cdot00013$.

Hence $z=231{\cdot}41$.
The error in x/z is less than

$$\frac{1}{10}\cdot\frac{1}{231}\cdot\frac{{\cdot}3835}{120000}=1{\cdot}5/10^9,$$

while the error in y/z will be found to be less than $4/10^9$. Hence we expect accuracy to eight decimal places, and we actually obtain

$$\log 2={\cdot}30103000,$$
$$\log 3={\cdot}47712125.$$

Finally, by using binomial coefficients, we can obtain logarithms to base $1{\cdot}001$; and it will be found that calculation by proportional parts gives common logarithms correct to seven decimal places, while the better approximation gives $\log_{10} 2$ and $\log_{10} 3$ correct to eleven decimal places. We have

$$(1{\cdot}001)^{12}=1{\cdot}012066220\ \dots,$$
$$(1{\cdot}001)^{23}=1{\cdot}023253+{\cdot}000001771+\dots,$$
$$(1{\cdot}001)^{40}=1{\cdot}040780+{\cdot}00000288+\dots.$$

With these facts we can proceed as before, the range for the proportional part to be avoided being still ${\cdot}36$ to ${\cdot}64$, since

$$\log_{1{\cdot}001} 10 > 2303{\cdot}6 \quad\text{and}\quad 2303{\cdot}6\times 1{\cdot}00018 > 2304,$$

by a very narrow margin, making the maximum error less than $5/10^8$.

Examples.

1. Show that by using the base 2 we obtain two-figure accuracy from the first upper bound.

2. Why do the proportional part equations

$$\log_{1{\cdot}1} 1{\cdot}25=2+\frac{{\cdot}4}{1{\cdot}21} \quad\text{and}\quad \log_{1{\cdot}1} 1{\cdot}28=2+\frac{{\cdot}7}{1{\cdot}21}$$

give the abnormally accurate result $\log_{10} 2={\cdot}301031$?

3. Investigate the degree of accuracy to be expected from using the base $1\frac{1}{4}$ in conjunction with proportional parts.

4. Give a geometrical interpretation of the first upper bound.

N. M. G.

GLEANINGS FAR AND NEAR.

1124. It is the story of the eternal triangle, the most useful of geometrical forms in the construction of a romantic pattern. Heigho! the trouble with human triangles is that they are never equilateral. Two sides together are invariably greater than the third.—Stacy Aumonier, *Old Iron*. [Per Mr. D. F. Ferguson.]

THE SOLUTION OF EQUATIONS BY THE USE OF PROPORTIONAL DIFFERENCES.

BY F. C. BOON.

NOWADAYS every Fifth Form pupil uses logarithm tables. A discussion of the construction of the difference columns, with graphical illustration, would prepare him for the methods to be described below, of an iterative process for the approximate solution of any equation. Incidentally it would help him to realise more fully than he usually does to what degree of accuracy his logarithmic work is reliable. The method of obtaining logarithms by using powers of 1·1, as described in the *Algebra Report*, uses the same principle of proportional parts, and, properly treated, helps to establish the general principle that, when the differences for unit increments in the entries used are constant, results obtained by proportional interpolation are accurate to the same number of figures as the entries.

Work on the above lines has a distinct educational value, and it seems desirable that it should be extended so that the pupil may realise that for any problem which he can reduce to an equation he can get a solution by successive approximation to n figures of accuracy if he possesses n-figure tables.

There are certain epidemic problems which from time to time are brought to mathematical teachers for solution.

Problem 1. *A goat is tethered to the circumference of a circular field; how long must his tether be (in terms of the radius of the field) if he can graze over half the field?*

Problem 2 concerns a ladder leaning against one wall and passing over another.

I have published elsewhere solutions of these problems, and shall choose variants for the illustrations of this article.

Problem 1 came my way many years ago. I tried various methods, including Newton's differentiation method, and eventually came to the conclusion that proportional interpolation had distinct advantages over the others. Here it is applied to a problem of the same type:—

given $$2\theta - \sin\theta = \tfrac{1}{2}\pi$$

say, $$f(\theta) = \tfrac{1}{2}\pi = 1{\cdot}5707963$$

to find a distance $(1-\cos\theta)$ inches.
(θ is, of course, in radian measure, but it will be convenient during the solution to give the values in degrees).

Now θ is clearly between 60° and 90°. Find $f(70°)$ and $f(80°)$.

$$f(70°) = 2{\cdot}4435 - {\cdot}9397 = 1{\cdot}5038,$$
$$f(80°) = 2{\cdot}7925 - {\cdot}9848 = 1{\cdot}8077.$$

Now $f(\theta)$ lies between these values, being 1·5708. Let $\theta = (70+h)°$.

Then by proportional differences,

$$\frac{(70+h)-70}{80-70} \simeq \frac{1{\cdot}5708-1{\cdot}5038}{1{\cdot}8077-1{\cdot}5038}$$

that is,
$$\frac{h^\circ}{10^\circ} \simeq \frac{\cdot0670}{\cdot3039},$$

and h is between 2 and 3. Try $f(72^\circ)$ and $f(73^\circ)$.

$$f(72^\circ)=2{\cdot}5133-{\cdot}9511=1{\cdot}5622,$$
$$f(73^\circ)=2{\cdot}5482-{\cdot}9563=1{\cdot}5919,$$

and
$$f(\theta), \text{ say, } f(72^\circ+k^\circ) \qquad =1{\cdot}5708.$$

Again, by proportional differences,

$$\frac{k}{1}=\frac{\cdot0086}{\cdot0297},$$

and k is apparently less than but nearly equal to ·3. Try $f(72^\circ\ 17')$ and $f(72^\circ\ 18')$.

$$f(72^\circ\ 17')=2{\cdot}5232-{\cdot}9526=1{\cdot}5706,$$
$$f(72^\circ\ 18')=2{\cdot}5238-{\cdot}9527=1{\cdot}5711,$$

and since in this part of the tables the differences are constant and $\frac{1}{2}\pi$ is nearly half-way between these results, the angle AOB is very nearly 144° 35′. Now the distance required is $(1-\cos\theta)$ inches. Tabulate :

	72° 17′	θ	72° 18′
$f(\theta)$ $(1-\cos\theta)$	1·5706 (1 − ·3040 − ·0003)	1·5708 (1 − ·3040 − x)	1·5711 (1 − ·3040)

By proportional differences,

$$\frac{\cdot0003}{\cdot0005}=\frac{x}{\cdot0003}$$

and
$$x=\cdot0002,$$

and the distance required is ·6958″ with the normal uncertainty about the fourth decimal place.

The solution of the equation has involved little working except looking up tables, and it is well within the powers of any Fifth Form pupil.

It is also to be noted that each successive approximation confirms the previous ones ; indeed, a slip at one stage is inevitably detected and corrected at the next. This makes for confidence and speed in working, and makes a final check unnecessary.

But an examination of 7-figure tables shows that in the neighbourhood of 72° the differences for 1′ in the values of $\sin\theta$, $\cos\theta$, and, of course, θ, are practically constant. Another step therefore gives a 7-figure solution.

$$f(72^\circ\ 17')=2{\cdot}5231643-{\cdot}9525730=1{\cdot}5705913,$$
$$f(72^\circ\ 18')=2{\cdot}5237461-{\cdot}9526615=1{\cdot}5710846\ ;$$

hence, proceeding as before,

	72° 17′	θ	72° 18′
$f(\theta)$	1·5705913	1·5707963	1·5710846
$\cos\theta$	·3043102	$\cdot 3040331 + x$	·3040331

and by proportional parts,

$$\frac{x}{\cdot 0002771} = \frac{\cdot 0002883}{\cdot 0004933},$$

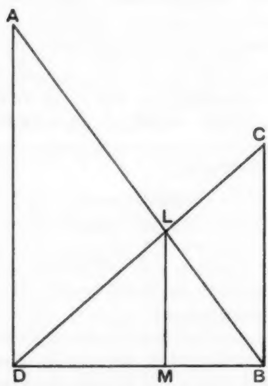
and $$x = \cdot 0001619,$$

and the distance required is

$$(1 - \cdot 3040331 - \cdot 0001619) = \cdot 6958050'',$$

with the normal uncertainty about the seventh place.

Problem 2. In this ladder problem *AB is a ladder 50′ long with its foot B against one wall of a corridor ; CD is another ladder 20′ long with its foot D against the other wall. They cross at L, 6′ above the floor. Find DB.*



This problem was brought to me by a colleague for a Sixth-Form boy a few weeks before the writing of this article, which it may be regarded as having provoked. For a boy ignorant of trigonometry, of the solution of a biquadratic and Horner's method, the following solution would have been possible.

Let $$AL/AB = DL/LC = x.$$

Then x is given by

$$x^2 - x^4 \quad \text{or} \quad x^2(1-x)(1+x) = 12(2x-1)/700 \,;$$

in either form, by the use of tables of squares or logarithms, the solution can be obtained.

For the Sixth-Form boy the following seemed preferable. It shows how to deal with a pair of simultaneous equations.

Let $\angle ABD=\theta, \quad \angle CDB=\phi.$

Then we have $DB=50 \cos \theta=20 \cos \phi,$

that is, $\cos \theta=0{\cdot}4 \cos \phi,$ ....(i)

and $$\frac{1}{LM}=\frac{1}{AD}+\frac{1}{CB};$$

that is, $$\frac{1}{6}=\frac{\operatorname{cosec} \theta}{50}+\frac{\operatorname{cosec} \phi}{20},$$

or $3\tfrac{1}{3}=0{\cdot}4 \operatorname{cosec} \theta+\operatorname{cosec} \phi=f(\phi)$, say. ....(ii)

From (ii), $\operatorname{cosec} \phi < 3\tfrac{1}{3} < \operatorname{cosec} 17^\circ\, 27',$

$$\phi > 17^\circ\, 27'.$$

Tabulate for trial. Using (i),

ϕ	20°	25°	30°
cos ϕ	·9397	·9063	·8660
cos θ	·3759	·3625	·3464
θ	67° 55′	68° 45′	69° 44′
cosec θ	1·0792	1·0730	1·0660
0·4 cosec θ	·4317	·4292	·4264
cosec ϕ	2·9238	2·3662	
$f(\phi)=$	3·3555		

It is here clear that ϕ is rather greater than 20°. Tabulate for $\phi=20°$ and $\phi=21°$.

ϕ	20°	21°	20° 10′	20° 8′
cos ϕ	·9397	·9336	·93869	·93889
cos θ	·3759	·3734	·37548	·37556
θ	67° 55′	68° 4′	67° 57′	67° 56′
cosec θ	1·0792	1·0780	1·07895	1·07898
0·4 cosec θ	·4317	·4312	·43158	·43159
cosec ϕ	2·9238	2·7904	2·90063	2·90524
$f(\phi)$	3·3555	3·2216	3·33221	3·33683

Letting $f(\phi^\circ + x') = 3{\cdot}3333$, proportional differences give

$$\frac{\cdot 0222}{\cdot 1339} = \frac{x}{60} \quad \text{and} \quad x \simeq 10.$$

Now, using five-figure tables, tabulate for $\phi = 20^\circ\ 10'$ and $20^\circ\ 8'$, being prepared to use extrapolation. The tabulation is given above, and the values for cosec θ were obtained thus :

θ	$67^\circ\ 56'$			$67^\circ\ 57'$
$\cos\theta$	·37569	·37556	·37548	·37542
cosec θ	1·07904			1·07892

The cosine and cosecant of $67^\circ\ 57'$ and $67^\circ\ 56'$ were taken from the tables, and the differences for $1'$ in the cosecants, being nearly half-differences for the cosine, the values for cosec θ were obtained more accurately by proportional parts than by assuming $67^\circ\ 57'$ and $67^\circ\ 56'$ to be sufficiently correct for arc cos ·37542 and arc cos ·37569. Then DB, namely, $50 \cos\phi$, can be obtained by proportional differences using

$f(\phi)$	3·33221	3·33333	3·33683
$\cos\phi$	·93869		·93889

It may happen that in the affairs of life a quadratic equation turns up which the Certificate candidate would hate to encounter in his examination, for example :

$$(1{\cdot}005 - r)^2 - (1{\cdot}393r + 0{\cdot}4237)^2 - (r + 0{\cdot}485)^2 = 0.$$

Early in the war, the engineering workshops of a London polytechnic undertook to make fuse adapters for shells. The geometrical specification (of which I have kept no note) gave the above equation for the radius, r inches, of the gauge to be used. The diameter was to be correct to 0·001″. Two solvers, of whom I was one, solved it by completing the square. Checking my result, an important matter in such a case, I found that I had slipped somewhere. The other solver's solution was also incorrect. I thereupon tried the " proportional difference " method, using tables of squares, felt confidence in each step, and found it speedier.

Writing the equation as $f(r) = 0$:

$$f(\cdot 1) \simeq \cdot 81 - \cdot 31 - \cdot 34 = \cdot 16,$$
$$f(\cdot 2) \simeq \cdot 64 - \cdot 49 - \cdot 47 = -\cdot 32.$$

This gives $r \simeq \cdot 133$.

$$\begin{aligned} f(\cdot 14) &= \cdot 865^2 - \cdot 6187^2 - \cdot 625^2 \\ &= \cdot 7482 - \cdot 3828 - \cdot 3906 = -\cdot 0252, \\ f(\cdot 13) &= \cdot 875^2 - \cdot 6048^2 - \cdot 615^2 \\ &= \cdot 7656 - \cdot 3658 - \cdot 3782 = \cdot 0216. \end{aligned}$$

This gives $r \simeq \cdot 1346$.

$$f(\cdot 1346) = \cdot 8704^2 - \cdot 6112^2 - \cdot 6196^2$$
$$= \cdot 7576 - \cdot 3735 - \cdot 3839 = \cdot 0002.$$

The result is, with the usual reservation, correct to four figures, and the diameter $= \cdot 269''$, correct to $\cdot 001''$.

Here, then, is something to round off the year's work during the lessons that remain after the Certificate examination. For many pupils it will be their " farewell to mathematics ". With no thought of preparation for examination, they will get an idea of the power and usefulness of mathematical method. The solution of equations need not for the rest of their lives be restricted to three formal categories.

In introducing the topic it may be advisable to begin with a revision of the work mentioned in the first paragraph of the article, to proceed to the solution of a fairly simple quadratic, then to a cubic such as $(x+1)(x+2)(x+3)=100$, treating it as

$$\log(x+1) + \log(x+2) + \log(x+3) = 2,$$

and then to any equation that fancy suggests and to some problems. It need hardly be said that graphical methods should be freely used, both to explain the principle and to get a first idea of the solution and the behaviour of the function involved. F. C. B.

1125. Look at the degraded terminology of mechanics—the very name being a misnomer with its so-called mechanical powers and other misleading and incorrect expressions. Any attempt again to talk scientifically of heat or the variations of temperature involves, on the now proved dynamical hypothesis, a series of misstatements—a string of verbal confusions. When we compare this with metaphysical terminology and its perfect adaptation to the various theories it has to express, we cannot help being painfully conscious of the incapacity of scientific men to deal with this really most important of all subjects.—R. L. Stevenson, *Memories and Portraits*, Tusitala edition, p. 174. [Per Mr. C. E. Kemp.]

1126. Scientific men, who imagine that their science affords an answer to the problems of existence, are perhaps the most to be pitied of mankind ; and contemned.—R. L. Stevenson, *Memories and Portraits*, Tusitala edition, p. 175. [Per Mr. C. E. Kemp.]

1127. I think the paradox about multiplying by nothing, was the first thing that overset and disgusted me in Algebra. It is simple enough if laboriously explained in words ; . . . .

Hence, briefly :

(1) " multiplied by one " means that a number is present in the question in hand ;

(2) " multiplied by two " means that this number and another number equal to it are both present in the question in hand ; while

(3) " multiplied by nothing " means simply that the number is absent.—R. L. Stevenson, *Memories and Portraits*, Tusitala edition, p. 181. [Per Mr. C. E. Kemp.]

THEOREMS ON THE TETRAHEDRON.

By R. T. Robinson.

1. *If a tetrahedron with its opposite edges perpendicular is inscribed in a conicoid, its orthocentre (the intersection of the perpendiculars from the vertices to the opposite faces) will lie on the conicoid if the asymptotic cone of the conicoid has three perpendicular generators.*

Taking the tetrahedron $ABCD$ as tetrahedron of reference, the equation of the conicoid is

$$u_1\beta\gamma+v_1\gamma\alpha+w_1\alpha\beta+r\alpha\delta+s\beta\delta+t\gamma\delta=0.$$

The coordinates of the orthocentre are

$$\left\{\frac{1}{A(b^2+c^2-a^2)},\ \frac{1}{B(a^2+c^2-b^2)},\ \frac{1}{C(a^2+b^2-c^2)},\ \frac{1}{D(e^2+f^2-a^2)}\right\}.$$

The orthocentre lies on the conicoid if

$$\begin{aligned}&u_1(b^2+c^2-a^2)(e^2+f^2-a^2)/BC+v_1(a^2+c^2-b^2)(e^2+f^2-a^2)/AC\\ &\quad+w_1(a^2+b^2-c^2)(e^2+f^2-a^2)/AB\\ &\quad+r(a^2+c^2-b^2)(a^2+b^2-c^2)/AD+s(b^2+c^2-a^2)(a^2+b^2-c^2)/BD\\ &\quad+t(b^2+c^2-a^2)(a^2+c^2-b^2)/CD=0,\end{aligned}$$

or $u_1\cos BC+v_1\cos AC+w_1\cos AB$

$$+r\cos AD+s\cos BD+t\cos CD=0.$$

But this is the condition that the asymptotic cone of the conicoid should have three perpendicular generators.

2. *If a conicoid is inscribed in a tetrahedron $ABCD$, the coordinates of the points of contact A_1, B_1, C_1, D_1 of the faces BCD, ... with the conicoid can be put in a simplified form.*

Since the faces of the tetrahedron $ABCD$ touch the conicoid, if the equation of the conicoid is

$$u\alpha^2+v\beta^2+w\gamma^2+\kappa\delta^2$$
$$+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta+2r\alpha\delta+2s\beta\delta+2t\gamma\delta=0$$

we get

$$1+\frac{2u_1st}{vw\kappa}-\frac{t^2}{w\kappa}-\frac{u_1^2}{vw}-\frac{s^2}{v\kappa}=0,$$

$$1+\frac{2w_1rs}{uv\kappa}-\frac{s^2}{v\kappa}-\frac{w_1^2}{uv}-\frac{r^2}{u\kappa}=0,$$

$$1+\frac{2v_1rt}{uw\kappa}-\frac{t^2}{w\kappa}-\frac{v_1^2}{uw}-\frac{r^2}{u\kappa}=0,$$

$$1+\frac{2u_1v_1w_1}{uvw}-\frac{u_1^2}{vw}-\frac{v_1^2}{uw}-\frac{w_1^2}{uv}=0.$$

Putting

$$\frac{u_1}{\sqrt{(vw)}}=x_1,\ \frac{v_1}{\sqrt{(uw)}}=x_2,\ \frac{w_1}{\sqrt{(uv)}}=x_3,\ \frac{r}{\sqrt{(u\kappa)}}=x_4,$$

$$\frac{s}{\sqrt{(v\kappa)}}=x_5,\ \frac{t}{\sqrt{(w\kappa)}}=x_6,$$

these equations become

$$\begin{gathered}1+2x_1x_5x_6-x_6^2-x_1^2-x_5^2=0,\\ 1+2x_3x_4x_5-x_5^2-x_3^2-x_4^2=0,\\ 1+2x_2x_4x_6-x_6^2-x_2^2-x_4^2=0,\\ 1+2x_1x_2x_3-x_1^2-x_2^2-x_3^2=0.\end{gathered}$$

These equations are satisfied by $x_4=x_1$, $x_5=x_2$, $x_6=x_3$, with the relation

$$1+2x_1x_2x_3-x_1^2-x_2^2-x_3^2=0.$$

They are also satisfied by $x_4=-x_1$, $x_5=-x_2$, $x_6=-x_3$, with the same relation.

Since the faces of $ABCD$ are tangent planes to the conicoid, the edges of the tetrahedron meet the conicoid in imaginary points. Where the edge BC meets the conicoid we have

$$v\beta^2+w\gamma^2+2u_1\beta\gamma=0.$$

Thus $u_1^2<vw$, that is, $x_1<1$, and similarly x_2, x_3, x_4, x_5, $x_6<1$.

If we put $x_1=\cos\alpha_1$, $x_2=\cos\alpha_2$, $x_3=\cos\alpha_3$, the relation

$$1+2x_1x_2x_3-x_1^2-x_2^2-x_3^2=0$$

becomes $\cos^2\alpha_1+\cos^2\alpha_2+\cos^2\alpha_3-2\cos\alpha_1\cos\alpha_2\cos\alpha_3=1,$

which reduces to

$$\sin\tfrac{1}{2}(\alpha_1+\alpha_2+\alpha_3)\sin\tfrac{1}{2}(\alpha_2+\alpha_3-\alpha_1)\sin\tfrac{1}{2}(\alpha_3+\alpha_1-\alpha_2)\sin\tfrac{1}{2}(\alpha_1+\alpha_2-\alpha_3)=0,$$

that is, $\alpha_1+\alpha_2+\alpha_3=2n\pi,$

or $\alpha_2+\alpha_3-\alpha_1=2n\pi,$

or $\alpha_3+\alpha_1-\alpha_2=2n\pi,$

or $\alpha_1+\alpha_2-\alpha_3=2n\pi.$

Thus if the faces of $ABCD$ touch the conicoid we must have

$$\frac{r^2}{u\kappa}=\frac{u_1^2}{vw};\quad \frac{s^2}{v\kappa}=\frac{v_1^2}{uw};\quad \frac{t^2}{w\kappa}=\frac{w_1^2}{uv};$$

and $uvw+2u_1v_1w_1-uu_1^2-vv_1^2-ww_1^2=0.$

If we take $\alpha_1+\alpha_2+\alpha_3=2n\pi$ and $x_4=x_1$, $x_5=x_2$, $x_6=x_3$, the coordinates of the points of contact A_1, B_1, C_1, D_1 can be written in a simplified form. The coordinates of A_1 are

$$\{0,\ (u_1t-ws),\ (u_1s-vt),\ (vw-u_1^2)\};$$

since $r/\sqrt{(u\kappa)}=u_1/\sqrt{vw})$, etc.,

$$\begin{aligned}u_1t-ws&=w\sqrt{(v\kappa)}\sin\alpha_1\sin\alpha_3,\\ u_1s-vt&=v\sqrt{(w\kappa)}\sin\alpha_1\sin\alpha_2,\\ vw-u_1^2&=vw\sin^2\alpha_1.\end{aligned}$$

Thus the coordinates of the four points are :

$$A_1 : \left\{0,\ \frac{\sin\alpha_3}{\sqrt{v}},\ \frac{\sin\alpha_2}{\sqrt{w}},\ \frac{\sin\alpha_1}{\sqrt{\kappa}}\right\};$$

$$B_1 : \left\{\frac{\sin\alpha_3}{\sqrt{u}},\ 0,\ \frac{\sin\alpha_1}{\sqrt{w}},\ \frac{\sin\alpha_2}{\sqrt{\kappa}}\right\};$$

$$C_1 : \left\{\frac{\sin\alpha_2}{\sqrt{u}},\ \frac{\sin\alpha_1}{\sqrt{v}},\ 0,\ \frac{\sin\alpha_3}{\sqrt{\kappa}}\right\};$$

$$D_1 : \left\{\frac{\sin\alpha_1}{\sqrt{u}},\ \frac{\sin\alpha_2}{\sqrt{v}},\ \frac{\sin\alpha_3}{\sqrt{w}},\ 0\right\}.$$

We get similar results for the seven other associated conicoids which touch the faces of the tetrahedron $ABCD$.

3. *For any one of the eight spheres inscribed in a tetrahedron $ABCD$, the opposite edges of the tetrahedron subtend equal or supplementary angles at the points of contact of the respective tangent planes through these edges.*

The equation of any sphere, centre O and radius ρ, is

$$\Sigma A\alpha \,.\, \Sigma A\alpha(AO^2-\rho^2)-\Sigma c^2 AB\alpha\beta=0.$$

If the sphere is an inscribed sphere and if T_1, T_2, T_3, T_4 are the tangents to the sphere from A, B, C, D respectively, the equation can be put in the form

$$\begin{aligned}&A^2T_1^2\alpha^2+B^2T_2^2\beta^2+C^2T_3^2\gamma^2+D^2T_4^2\delta^2\\&\quad+AB\alpha\beta(T_1^2+T_2^2-c^2)+AC\alpha\gamma(T_1^2+T_3^2-b^2)\\&\quad+BC\beta\gamma(T_2^2+T_3^2-a^2)+AD\alpha\delta(T_1^2+T_4^2-d^2)+\ldots=0.\end{aligned}$$

But if a conicoid is inscribed in a tetrahedron,

$$\frac{r^2}{u\kappa}=\frac{u_1^2}{vw};\quad \frac{s^2}{v\kappa}=\frac{v_1^2}{uw};\quad \frac{t^2}{w\kappa}=\frac{w_1^2}{uv}.$$

Thus for an inscribed sphere

$$\frac{(T_1^2+T_4^2-d^2)^2}{T_1^2T_4^2}=\frac{(T_2^2+T_3^2-a^2)^2}{T_2^2T_3^2}$$

and two similar equations, or

$$\frac{T_1^2+T_4^2-d^2}{T_1T_4}=\pm\frac{T_2^2+T_3^2-a^2}{T_2T_3};$$

but the expression on the left is the cosine of the angle subtended by AD at the points of contact of the tangent planes through AD, and that on the right is the cosine of the angle subtended by BC at the points of contact of the tangent planes through BC. Hence the theorem.

It is to be noted that if the coefficient of any one of the terms $\alpha\beta$, $\beta\gamma$, ... , say, that of $\alpha\beta$, is zero, the corresponding edge of the

tetrahedron, in this case AB, subtends a right angle at the points of contact of the tangent planes through AB to the sphere.

The equation of the self-conjugate sphere of the tetrahedron $ABCD$ is

$$A^2(b^2+c^2-a^2)\alpha^2+B^2(a^2+c^2-b^2)\beta^2+C^2(a^2+b^2-c^2)\gamma^2 \\ +D^2(e^2+f^2-a^2)\delta^2=0.$$

Hence *any edge of the tetrahedron ABCD subtends a right angle at the points of contact of the tangent planes through the edge to the self-conjugate sphere*—a result which is easily seen geometrically.

4. *If a conicoid is inscribed in a tetrahedron ABCD,* A_1, B_1, C_1, D_1 *being the points of contact of the faces BCD, ACD, ... then*

$$B_1C_1(AA_1DD_1)=A_1D_1(BB_1CC_1),$$
$$A_1C_1(BB_1DD_1)=B_1D_1(AA_1CC_1).$$
$$C_1D_1(AA_1BB_1)=A_1B_1(CC_1DD_1).$$

Let the equation of the conicoid be

$$u\alpha^2+v\beta^2+w\gamma^2+\kappa\delta^2 \\ +2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta+2r\alpha\delta+2s\beta\delta+2t\gamma\delta=0.$$

Let the polar $A_1B_1C_1$ of D meet AD at F. The equation of the polar of D is

$$r\alpha+s\beta+t\gamma+\kappa\delta=0,$$

and where this meets $AD\,(\beta=0,\ \gamma=0)$

$$\alpha/\delta=-\kappa/r,$$

so that $A\alpha/D\delta=-A\kappa/Dr$ is $FD/AF=-A\kappa/Dr$.

Let the polar $B_1C_1D_1$ of A meet AD at E. Then similarly

$$DE/AE=-Ar/Du.$$

Thus
$$\frac{DE\,.\,AF}{AE\,.\,FD}=\frac{r^2}{u\kappa}.$$

Similarly if the polars of B and C meet BC at K and L respectively,

$$\frac{CK\,.\,BL}{KB\,.\,CL}=\frac{u_1^2}{vw}.$$

But
$$r^2/u\kappa=u_1^2/vw.$$

Hence $(DEAF)=(CKBL)$ or $(AFDE)=(BLCK)$.

Thus
$$B_1C_1(AFDE)=A_1D_1(BLCK),$$
or
$$B_1C_1(AA_1DD_1)=A_1D_1(BB_1CC_1),$$
and similarly for the other two results of the theorem.

5. *To find the radius* ρ *and the quadriplanar coordinates of the centre* Ω *of a sphere which touches internally the inscribed sphere, centre* I, *radius* r, *and touches externally the escribed spheres, centres* I_1, I_2, I_3, I_4, *radii* r_1, r_2, r_3, r_4 *of the tetrahedron ABCD when that is possible.*

PRELIMINARY.

Taking three rectangular axes through the centre O of the sphere $ABCD$, and (a, b, c), (a_1, b_1, c_1) ... as the coordinates of I, I_1, ... the centre of the sphere must satisfy the equation

$$(x-a)^2+(y-b)^2+(z-c)^2-(\rho-r)^2=0$$

and four equations of the type

$$(x-a_1)^2+(y-b_1)^2+(z-c_1)^2-(\rho+r_1)^2=0.$$

If T, T_1, T_2, T_3, T_4 are the lengths of the tangents from O to the inscribed and escribed spheres, these equations can be written

$$x^2+y^2+z^2-2ax-2by-2cz+T^2-\rho^2+2\rho r=0\equiv S, \quad \text{......(i)}$$

$$x^2+y^2+z^2-2a_1x-2b_1y-2c_1z+T_1^2-\rho^2-2\rho r_1=0\equiv S_1, \quad \text{.....(ii)}$$

$$S_2=0, \quad S_3=0, \quad S_4=0. \quad \text{............(iii), (iv), (v)}$$

If (x, y, z) satisfies these five equations, then multiplying (i)-(v) by $-2/r$, $1/r_1$, $1/r_2$, $1/r_3$, $1/r_4$ and adding, we get

$$(x^2+y^2+z^2-\rho^2)\left(-\frac{2}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)$$

$$-2x\left(-\frac{2a}{r}+\frac{a_1}{r_1}+\frac{a_2}{r_2}+\frac{a_3}{r_3}+\frac{a_4}{r_4}\right)+\ldots+\left(\sum\frac{T_1^2}{r_1}-\frac{2T^2}{r}\right)-12\rho=0\,;$$

but

$$\frac{2}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4},$$

and I is the centroid of masses $1/r_1$ at I_1, $1/r_2$ at I_2, ... ,

hence

$$a=\left(\sum\frac{a_1}{r_1}\right)\Big/\left(\sum\frac{1}{r_1}\right)=\frac{r}{2}\sum\frac{a_1}{r_1}$$

or

$$\frac{2a}{r}=\sum\frac{a_1}{r_1}.$$

Similarly

$$\frac{2b}{r}=\sum\frac{b_1}{r_1} \text{ and } \frac{2c}{r}=\sum\frac{c_1}{r_1}.$$

Thus the last equation becomes

$$\rho=\frac{1}{12}\left\{\sum\frac{T_1^2}{r_1}-\frac{2T^2}{r}\right\}.$$

Thus if ρ has this value and (x, y, z) satisfies the equations $S=0$, $S_1=0$ ... ,

$$-\frac{2S}{r}+\frac{S_1}{r_1}+\frac{S_2}{r_2}+\frac{S_3}{r_3}+\frac{S_4}{r_4}=0,$$

that is, for this value of ρ if S_1, S_2, S_3, S_4 have a common point, this point must lie on $S=0$.

The equations of the radical planes of S, S_1; S, S_2; ... are

$$2(a-a_1)x+2(b-b_1)y+2(c-c_1)z+T_1^2-T^2-2\rho(r_1+r)=0,$$

$$2(a-a_2)x+2(b-b_2)y+2(c-c_2)z+T_2^2-T^2-2\rho(r_2+r)=0,$$

or $$2(a-a_1)x+2(b-b_1)y+2(c-c_1)z=-L_1r_1, \quad \text{............(vi)}$$
and three similar equations, where $L_1 \equiv \{T_1^2-T^2-2\rho(r+r_1)\}/r_1$, etc.,
and $$L_1+L_2+L_3+L_4=0.$$

If these equations are multiplied by $1/r_1$, $1/r_2$, $1/r_3$, $1/r_4$ and the resulting equations added, the coefficients of x, y and z and the constant term are all zero. Thus the four radical planes of S, S_1; S, S_2; S, S_3; S, S_4 meet in a point. This point is got by solving any three of the four equations typified by (vi).

If the coordinates of this point satisfy one of the equations (i)-(v), they satisfy the remaining four, and in this case there is a sphere which touches internally the inscribed sphere and touches externally the escribed sphere's centres I_1, I_2, I_3, I_4, its radius being
$$\rho=\frac{1}{12}\left\{\sum\frac{T_1^2}{r_1}-\frac{2T^2}{r}\right\}:$$

The coordinates of the centre are linear functions of L_1, L_2, L_3, L_4, and if the plane of YZ is parallel to the face BCD of the tetrahedron it will be seen from the sequel that
$$x=\tfrac{1}{4}(-L_1+L_2\cos AB+L_3\cos AC+L_4\cos AD).$$

If, using quadriplanar coordinates, the same method is adopted in the case of a tetrahedron $ABCD$, its inscribed sphere and the four escribed spheres opposite to A, B, C, D and if we assume that a sphere can be described touching these five spheres, the inscribed sphere being inside and the four escribed spheres outside, and if ρ is the radius of the assumed sphere and $(\alpha, \beta, \gamma, \delta)$ the coordinates of its centre, the tetrahedron of reference being $ABCD$, then
$$\begin{aligned}-9V^2(\rho-r)^2&=\Sigma c^2AB(\alpha-r)(\beta-r)\\&=\Sigma c^2AB\alpha\beta+r^2\Sigma c^2AB-r\alpha(c^2AB+b^2AC+d^2AD)\ldots,\end{aligned}$$
or
$$\begin{aligned}&\alpha(c^2AB+b^2AC+d^2AD)+\beta(c^2AB+a^2BC+e^2BD)\\&\qquad+\gamma(b^2AC+a^2BC+f^2CD)\\&+\delta(d^2AD+e^2BD+f^2CD)-(\Sigma c^2AB\alpha\beta)/r\\&\qquad=r\Sigma c^2AB+9V^2(\rho-r)^2/r. \qquad \text{(i)}\end{aligned}$$

For the escribed spheres we have
$$\begin{aligned}&\alpha(c^2AB+b^2AC+d^2AD)+\beta(-c^2AB+a^2BC+e^2BD)\\&\qquad+\gamma(-b^2AC+a^2BC+f^2CD)\\&+\delta(-d^2AD+e^2BD+f^2CD)-(\Sigma c^2AB\alpha\beta)/r_1\\&=r_1(a^2BC+e^2BD+f^2CD-c^2AB-b^2AC-d^2AD)\\&\qquad+9V^2(\rho+r_1)^2/r;\ \text{......................(ii)}\end{aligned}$$
$$\begin{aligned}&\alpha(-c^2AB+b^2AC+d^2AD)+\beta(c^2AB+a^2BC+e^2BD)\\&\qquad+\gamma(b^2AC-a^2BC+f^2CD)\\&+\delta(d^2AD-e^2BD+f^2CD)-(\Sigma c^2AB\alpha\beta)/r_2\\&=r_2(-a^2BC-e^2BD+f^2CD-c^2AB+b^2AC+d^2AD)\\&\qquad+9V^2(\rho+r_2)^2/r\ldots, \quad \text{......(iii)}\end{aligned}$$
and two similar equations (iv) and (v).

N

We have five equations and there are five unknown quantities, $\alpha, \beta, \gamma, \delta, \rho$ connected by the relation $\Sigma A\alpha = 3V$.

Assuming that these equations are simultaneously true, adding the equations (ii)-(v) and subtracting the result from twice the equation (i) and using the relation

$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4},$$

we get

$$\begin{aligned} O = 2r\Sigma c^2AB &+ a^2BC(-r_1+r_2+r_3-r_4) + e^2BD(-r_1+r_2-r_3+r_4) \\ &+ f^2CD(-r_1-r_2+r_3+r_4) + c^2AB(r_1+r_2-r_3-r_4) \\ &\qquad\qquad + b^2AC(r_1-r_2+r_3-r_4) \\ &+ d^2AD(r_1-r_2-r_3+r_4) - 108V^2\rho + 9V^2(2r - \Sigma r_1). \end{aligned}$$

If we write $P \equiv c^2AB + b^2AC + d^2AD, \quad Q \equiv c^2AB + a^2BC + e^2BD,$

$R \equiv b^2AC + a^2BC + f^2CD, \quad S \equiv d^2AD + e^2BD + f^2CD,$

$X \equiv \Sigma c^2AB, \quad$ where $\quad P + Q + R + S = 2X,$

it can be proved that

$$\begin{aligned} IO_1^2 &= \rho_1^2 - Xr^2/9V^2, \\ I_1O_1^2 &= \rho_1^2 - (X - 2P)r_1^2/9V^2, \\ I_2O_1^2 &= \rho_1^2 - (X - 2Q)r_2^2/9V^2, \text{ etc.,} \end{aligned}$$

where O_1 is the centre and ρ_1 the radius of the sphere $ABCD$.

Thus the equation for ρ becomes

$$\begin{aligned} 108V^2\rho &= 2r(X + 9V^2) - 9V^2\Sigma r_1 - r_1(X - 2P) - r_2(X - 2Q) \ldots, \\ &= 9V^2\left\{\sum \frac{I_1O_1^2 - r_1^2}{r_1} - \frac{2(IO_1^2 - r^2)}{r}\right\} + 9V^2\rho_1^2\left(\frac{2}{r} - \sum\frac{1}{r_1}\right) \\ &= 9V^2\left\{\sum\frac{T_1^2}{r_1} - \frac{2T^2}{r}\right\}, \end{aligned}$$

or

$$\rho = \frac{1}{12}\left\{\sum\frac{T_1^2}{r_1} - \frac{2T^2}{r}\right\}.$$

Calling the spheres whose equations are (i), (ii), ... $S, S_1, \ldots$ and multiplying (i) by r, (ii) by r_1, and so on, we get for the equation of the radical plane of S and S_1,

$$\begin{aligned} P\alpha(r - r_1) &+ Q\beta\{r - (Q - 2c^2AB)r_1/Q\} + \ldots \\ &= Xr^2 - (X - 2P)r_1^2 + 9V^2\{(\rho - r)^2 - (\rho + r_1)^2\} \\ &= 9V^2(\rho_1^2 - IO_1^2) + 9V^2(I_1O_1^2 - \rho_1^2) \\ &\qquad\qquad + 9V^2\{-2\rho(r + r_1) + r^2 - r_1^2\} \\ &= 9V^2\{T_1^2 - T^2 - 2\rho(r + r_1)\} \\ &= 9V^2 . L_1r_1, \end{aligned}$$

or $$P\alpha\left(\frac{r-r_1}{r_1}\right)+Q\beta\left(\frac{r-r_1}{r_1}+\frac{2c^2AB}{Q}\right)+R\gamma\left(\frac{r-r_1}{r}+\frac{2b^2AC}{R}\right)$$
$$+S\delta\left(\frac{r-r_1}{r_1}+\frac{2d^2AD}{S}\right)=9V^2L_1. \quad \text{............(vi)}$$

Similarly the radical planes of S, S_2; S, S_3; S, S_4 are

$$P\alpha\left(\frac{r-r_2}{r_2}+\frac{2c^2AB}{P}\right)+Q\beta\left(\frac{r-r_2}{r_2}\right)+R\gamma\left(\frac{r-r_2}{r_2}+\frac{2a^2BC}{R}\right)$$
$$+S\delta\left(\frac{r-r_2}{r_2}+\frac{2e^2BD}{S}\right)=9V^2L_2, \quad \text{............(vii)}$$

$$P\alpha\left(\frac{r-r_3}{r_3}+\frac{2b^2AC}{P}\right)+Q\beta\left(\frac{r-r_3}{r_3}+\frac{2a^2BC}{Q}\right)+R\gamma\left(\frac{r-r_3}{r_3}\right)$$
$$+S\delta\left(\frac{r-r_3}{r_3}+\frac{2f^2CD}{S}\right)=9V^2L_3, \quad \text{...........(viii)}$$

$$P\alpha\left(\frac{r-r_4}{r_4}+\frac{2d^2AD}{P}\right)+Q\beta\left(\frac{r-r_4}{r_4}+\frac{2e^2BD}{Q}\right)+R\gamma\left(\frac{r-r_4}{r_4}+\frac{2f^2CD}{R}\right)$$
$$+S\delta\left(\frac{r-r_4}{r_4}\right)=9V^2L_4. \quad \text{........................(ix)}$$

If these four equations are added the coefficients of $\alpha, \beta, \gamma, \delta$ are zero and so is the right-hand side, since $L_1+L_2+L_3+L_4=0$. Thus the radical planes of S, S_1; S, S_2; S, S_3; S, S_4 meet in a point, and to find $\alpha, \beta, \gamma, \delta$ three of these equations can be taken with the equation
$$A\alpha+B\beta+C\gamma+D\delta=3V.$$

If we take this equation with (vi), (vii) and (viii), we get

$$\alpha PQRS\begin{vmatrix} \frac{A}{P}, & \frac{B}{Q}, & \frac{C}{R}, & \frac{D}{S} \\ \frac{r-r_1}{r_1}, & \frac{r-r_1}{r_1}+\frac{2c^2AB}{Q}, & \vdots & \vdots \\ \frac{r-r_2}{r_2}+\frac{2c^2AB}{P}, & \frac{r-r_2}{r_2}, & \vdots & \vdots \\ \frac{r-r_3}{r_3}+\frac{2b^2AC}{P}, & \frac{r-r_3}{r_3}+\frac{2a^2BC}{Q}, & \vdots & \vdots \end{vmatrix}$$
$$=QRS\begin{vmatrix} 3V, & B/Q, & C/R, & D/S \\ 9V^2L_1, & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 9V^2L_2, & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 9V^2L_3, & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix}.$$

Writing $\alpha_1, \beta_1, \gamma_1$ for $(r-r_1)/r_1$, etc., in the determinant on the left-hand side of the equation the coefficient of A/P is

$$\left(\alpha_1+\frac{2c^2AB}{Q}\right)\left(\frac{2f^2CD\beta_1}{S}+\frac{2a^2BC\gamma_1}{R}-\frac{2e^2BD\gamma_1}{S}+\frac{4a^2f^2BC^2D}{RS}\right)$$
$$-\left(\alpha_1+\frac{2b^2AC}{R}\right)\left(\frac{2f^2CD\beta_1}{S}-\frac{2a^2BC\beta_1}{Q}-\frac{2e^2BD\gamma_1}{S}-\frac{4a^2e^2B^2CD}{QS}\right)$$
$$+\left(\alpha_1+\frac{2d^2AD}{S}\right)\left(-\frac{2a^2BC\beta_1}{Q}-\frac{2a^2BC\gamma_1}{R}-\frac{4a^4B^2C^2}{QR}\right)$$
$$=\alpha_1\left(\frac{4a^2f^2BC^2D}{RS}+\frac{4a^2e^2B^2CD}{QS}-\frac{4a^4B^2C^2}{QR}\right)$$
$$+\beta_1\left(\frac{4c^2f^2ABCD}{QS}-\frac{4b^2f^2AC^2D}{RS}-\frac{4a^2d^2ABCD}{QS}+\frac{4a^2b^2ABC^2}{QR}\right)$$
$$+\gamma_1\left(\frac{4a^2c^2AB^2C}{QR}-\frac{4c^2e^2AB^2D}{QS}+\frac{4b^2e^2ABCD}{RS}-\frac{4a^2d^2ABCD}{RS}\right)$$
$$+8a^2(b^2e^2+c^2f^2-a^2d^2)\left(\frac{AB^2C^2D}{QRS}\right).$$

The coefficient of α_1/QRS is

$$4a^2f^2BC^2D(c^2AB+a^2BC+e^2BD)$$
$$+4a^2e^2B^2CD(b^2AC+a^2BC+f^2CD)$$
$$-4a^4B^2C^2(d^2AD+e^2BD+f^2CD)$$
$$=4a^2(b^2e^2+c^2f^2-a^2d^2)AB^2C^2D+8a^2e^2f^2B^2C^2D^2.$$

The coefficient of β_1/QRS is

$$4a^2(b^2e^2+c^2f^2-a^2d^2)AB^2C^2D-4f^2(a^2d^2+b^2e^2-c^2f^2)ABC^2D^2.$$

The coefficient of γ_1/QRS is

$$4a^2(b^2e^2+c^2f^2-a^2d^2)AB^2C^2D-4e^2(a^2d^2+c^2f^2-b^2e^2)AB^2CD^2.$$

Noting that α_1 β_1, γ_1 are respectively equal to $-2A/\Sigma A$, $-2B/\Sigma A$, $-2C/\Sigma A$, the coefficient of A/P is

$$\frac{4AB^2C^2D^2}{QRS\Sigma A}\{2a^2(b^2e^2+c^2f^2-a^2d^2)$$
$$+2f^2(a^2d^2+b^2e^2-c^2f^2)+2e^2(a^2d^2+c^2f^2-b^2e^2)-4a^2e^2f^2\}$$
$$=\frac{4AB^2C^2D^2}{QRS\Sigma A}\,.\,576VV_1.$$

In the same determinant the coefficient of $-(r-r_1)/r_1$, the second term in the first column, is

$$\beta_1\left(\frac{2f^2BCD}{QS}-\frac{2f^2C^2D}{RS}+\frac{2a^2BC^2}{QR}-\frac{2a^2BCD}{QS}\right)$$
$$+\gamma_1\left(\frac{2a^2B^2C}{QR}-\frac{2e^2B^2D}{QS}+\frac{2e^2BCD}{RS}-\frac{2a^2BCD}{RS}\right)$$
$$+4a^2(e^2+f^2-a^2)\frac{B^2C^2D}{QRS}$$

$$=\frac{2\beta_1}{QRS}[ABC^2D\{a^2d^2+b^2e^2-c^2f^2-b^2(a^2+e^2-f^2)\}$$
$$+a^2(e^2+f^2-a^2)B^2C^2D-f^2(a^2+e^2-f^2)BC^2D^2]$$
$$+\frac{2\gamma_1}{QRS}[AB^2CD\{a^2d^2+c^2f^2-b^2e^2-c^2(a^2+f^2-e^2)\}$$
$$+a^2(e^2+f^2-a^2)B^2C^2D-e^2(a^2+f^2-e^2)B^2CD^2]$$
$$+4a^2(e^2+f^2-a^2)\frac{B^2C^2D}{QRS}$$
$$=\frac{4AB^2C^2D}{QRS\Sigma A}\{-2a^2d^2+b^2(a^2+e^2-f^2)+c^2(a^2+f^2-e^2)\}$$
$$+4a^2(e^2+f^2-a^2)\left(1-\frac{B+C}{\Sigma A}\right)\frac{B^2C^2D}{QRS}+\frac{4B^2C^2D^2}{QRS\Sigma A}$$
$$\{16A^2-a^2(e^2+f^2-a^2)\}$$
$$=\frac{4AB^2C^2D}{QRS\Sigma A}\{16AD\cos AD-a^2(e^2+f^2-a^2)\}+64\frac{A^2B^2C^2D^2}{QRS\Sigma A}$$
$$+4a^2(e^2+f^2-a^2)\left(1-\frac{B+C+D}{\Sigma A}\right)\frac{B^2C^2D}{QRS}$$
$$=64A^2B^2C^2D^2(1+\cos AD)/QRS\Sigma A.$$

In the same determinant the coefficient of $\left(\beta_1+\dfrac{2c^2AB}{P}\right)$, the third term in the first column, is

$$\alpha_1\left(\frac{2f^2BCD}{QS}-\frac{2f^2C^2D}{RS}+\frac{2a^2BC^2}{QR}-\frac{2a^2BCD}{QS}\right)$$
$$+\gamma_1\left(\frac{2b^2ABC}{QR}-\frac{2d^2ABD}{QS}-\frac{2c^2ABC}{QR}+\frac{2d^2ACD}{RS}+\frac{2c^2ABD}{QS}-\frac{2b^2ACD}{RS}\right)$$
$$+\frac{4ABC^2D}{QRS}\{a^2d^2+b^2e^2-c^2f^2-b^2(a^2+e^2-f^2)\}$$
$$=2\alpha_1[ABC^2D\{a^2d^2+b^2e^2-c^2f^2-b^2(a^2+e^2-f^2)\}$$
$$+a^2(e^2+f^2-a^2)B^2C^2D-f^2(a^2+e^2-f^2)BC^2D^2]/QRS$$
$$+2\gamma_1[AB^2CD\{b^2e^2+c^2f^2-a^2d^2-c^2(e^2+f^2-a^2)\}$$
$$+ABC^2D\{a^2d^2+b^2e^2-c^2f^2-b^2(a^2+e^2-f^2)\}$$
$$+ABCD^2\{a^2d^2-b^2e^2+c^2f^2-d^2(a^2+f^2-e^2)\}]/QRS$$
$$+\frac{4ABC^2D}{QRS}\{a^2d^2+b^2e^2-c^2f^2-b^2(a^2+e^2-f^2)\}$$
$$=-\frac{4A^2BC^2D}{QRS\Sigma A}\{a^2d^2+b^2e^2-c^2f^2-b^2(a^2+e^2-f^2)\}$$
$$+\frac{4AB^2C^2D}{QRS\Sigma A}\{-a^2(e^2+f^2-a^2)+c^2(e^2+f^2-a^2)+a^2d^2-b^2e^2-c^2f^2\}$$

$$+\frac{4ABC^2D^2}{QRS\Sigma A}\{f^2(a^2+e^2-f^2)+b^2e^2-a^2d^2-c^2f^2+d^2(a^2+f^2-e^2)\}$$

$$-\frac{4ABC^3D}{QRS\Sigma A}\{a^2d^2+b^2e^2-c^2f^2-b^2(a^2+e^2-f^2)\}$$

$$+\frac{4ABC^2D}{QRS\Sigma A}\{a^2d^2+b^2e^2-c^2f^2-b^2(a^2+e^2-f^2)\}.\Sigma A$$

$$=\frac{4AB^2C^2D}{QRS\Sigma A}\{2a^2d^2-2c^2f^2-a^2(e^2+f^2-a^2)$$
$$+c^2(e^2+f^2-a^2)-b^2(a^2+e^2-f^2)\}$$

$$+\frac{4ABC^2D^2}{QRS\Sigma A}\{2b^2e^2-2c^2f^2+f^2(a^2+e^2-f^2)$$
$$+d^2(a^2+f^2-e^2)-b^2(a^2+e^2-f^2)\}$$

$$=-\frac{64A^2B^2C^2D^2}{QRS\Sigma A}(\cos AD-\cos AB).$$

Similarly, in the same determinant the coefficient of

$$-\left(\gamma+\frac{2b^2AC}{P}\right),$$

the fourth term in the first column, is

$$\frac{64A^2B^2C^2D^2}{QRS\Sigma A}(\cos AD-\cos AC).$$

Hence the determinant which is the coefficient of $\alpha PQRS$ is

$$4.576\frac{A^2B^2C^2D^2}{PQRS\Sigma A}VV_1+\frac{2A}{\Sigma A}\cdot\frac{64A^2B^2C^2D^2}{QRS\Sigma A}(1+\cos AD)$$
$$+\left(-\frac{2B}{\Sigma A}+\frac{2c^2AB}{P}\right)\cdot\frac{64A^2B^2C^2D^2}{QRS\Sigma A}(\cos AB-\cos AD)$$
$$+\left(\frac{2C}{\Sigma A}-\frac{2b^2AC}{P}\right)\cdot\frac{64A^2B^2C^2D^2}{QRS\Sigma A}(\cos AD-\cos AC)$$

$$=4.576\frac{A^2B^2C^2D^2}{PQRS\Sigma A}VV_1$$
$$+128\frac{A^2B^2C^2D^2}{QRS\Sigma A}\left[\frac{1}{\Sigma A}\right.$$
$$\{A+A\cos AD+B\cos AD-B\cos AB+C\cos AD-C\cos AC\}$$
$$\left.+\frac{1}{P}\{c^2AB\cos AB+b^2AC\cos AC-\cos AD(P-d^2AD)\}\right]$$

$$=4.576\frac{A^2B^2C^2D^2}{PQRS\Sigma A}VV_1+128\frac{A^2B^2C^2D^2}{QRS\Sigma A}$$
$$[\cos AD+\{18(V^2-VV_1)-P\cos AD\}/P]$$

$$=4\,.\,576\,\frac{A^2B^2C^2D^2}{PQRS\Sigma A}\,VV_1+4\,.\,576\,\frac{A^2B^2C^2D^2}{PQRS\Sigma A}\,(V^2-VV_1)$$

$$=4\,.\,576\,\frac{A^2B^2C^2D^2}{PQRS}\cdot\frac{V^2}{\Sigma A}.$$

Thus the coefficient of α in the equation is $4\,.\,576A^2B^2C^2D^2V^2/\Sigma A$, and the right-hand side of the equation is

$$QRS\Big[3V\,.\,\frac{4A^2B^2C^2D^2}{QRS\Sigma A}\,.\,576VV_1-9V^2L_1\,.\,\frac{64A^2B^2C^2D^2}{QRS\Sigma A}\,(1+\cos AD)$$

$$+9V^2L_2\,.\,\frac{64A^2B^2C^2D^2}{QRS\Sigma A}(\cos AB-\cos AD)$$

$$+9V^2L_3\,.\,\frac{64A^2B^2C^2D^2}{QRS\Sigma A}(\cos AC-\cos AD)\Big]$$

$$=\frac{9V^2.\,64A^2B^2C^2D^2}{\Sigma A}$$

$$\left[\frac{12V_1}{A}-L_1+L_2\cos AB+L_3\cos AC-(L_1+L_2+L_3)\cos AD\right],$$

but $$L_1+L_2+L_3+L_4=0.$$

Thus

$$\alpha=\tfrac{1}{4}(-L_1+L_2\cos AB+L_3\cos AC+L_4\cos AD)+3V_1/A.$$

Similarly

$$\beta=\tfrac{1}{4}(L_1\cos BA-L_2+L_3\cos BC+L_4\cos BD)+3V_2/B,$$

and so on. The equation of the sphere which has this point for its centre and ρ for its radius can be put in the form

$$\Sigma A\alpha[\tfrac{3}{2}V(L_1\alpha+L_2\beta+L_3\gamma+L_4\delta)+(x^2-\rho^2)\Sigma A\alpha]-\Sigma c^2AB\alpha\beta=0, \quad \text{(x)}$$

where $$x^2\equiv\rho_1^2-\tfrac{3}{2}\Sigma(L_1V_1/A)+\tfrac{1}{16}(\Sigma L_1^2-2\Sigma L_1L_2\cos AB),$$

and x is the radius of the sphere whose equation is

$$\Sigma A\alpha\,.\,\Sigma(\tfrac{3}{2}VL_1\alpha)-\Sigma c^2AB\alpha\beta=0.$$

The sphere whose equation is (x) will touch the five inscribed spheres S, S_1, S_2, S_3, S_4 if its centre lies on S, and if its centre lies on S it also lies on the other four. The condition that the centre should lie on S is

$$x^2=\rho_1^2-IO_1^2+(\rho-r)^2,$$

for the equation of S can be written

$$\Sigma P\alpha-\frac{1}{r}\Sigma c^2AB\alpha\beta=r\Sigma c^2AB+\frac{9V^2}{r}(\rho-r)^2.$$

Substituting the coordinates of the centre in this equation, the right-hand side

$$=3\Sigma(PV_1/A)+\tfrac{1}{4}\{L_1(-P+Q\cos AB+R\cos AC+S\cos AB)$$
$$+L_2(P\cos BA-Q+R\cos BC+S\cos BD)\}$$
$$-\frac{9V^2\rho_1^2}{r}+\frac{9V^2}{16r}\{\Sigma L_1^2-2\Sigma L_1L_2\cos AB\}$$
$$=6V\rho_1^2\Sigma A-\tfrac{9}{2}V\Sigma A\Sigma(L_1V_1/A)-9V^2\rho_1^2/r$$
$$+9V^2[\Sigma L_1^2-2\Sigma L_1L_2\cos AB]/16r$$
$$=\frac{9V^2}{r}[\rho_1^2-\tfrac{3}{2}\Sigma(L_1V_1/A)+\tfrac{1}{16}(\Sigma L_1^2-2\Sigma L_1L_2\cos AB)]=9V^2x^2/r,$$

and the right-hand side of the equation is

$$\frac{9V^2}{r}(\rho_1^2-IO_1^2)+\frac{9V^2}{r}(\rho-r)^2.$$

Thus the result of the substitution is

$$x^2=\rho_1^2-IO_1^2+(\rho-r)^2.$$

It may be noted that in the case of a triangle ABC, the circle, the nine-points circle of the triangle, which touches the four inscribed circles has its radius $\frac{1}{2}R$ and

$$\frac{1}{2}R=\frac{1}{8}\left[\sum\frac{T_1^2}{r_1}-\frac{2T^2}{r}\right],$$

where T, T_1, T_2, T_3 are the lengths of the tangents from the centre of the circle ABC to the four inscribed circles and the coordinates of the centre of the nine-points circle can be put in the form :

$$\alpha=R\cos A+\tfrac{1}{4}(-L_1+L_2\cos C+L_3\cos B),$$
$$\beta=R\cos B+\tfrac{1}{4}(L_1\cos C-L_2+L_3\cos A),$$
$$\gamma=R\cos C+\tfrac{1}{4}(L_1\cos B+L_2\cos A-L_3),$$

where $\qquad L_1\equiv\{T_1^2-T^2-R(r+r_1)\}/r_1$, etc.

The three remaining escribed spheres centres

$$I_5(r_1, r_2, -r_3, -r_4),\ I_6(r_1, -r_2, r_3, -r_4) \text{ and } I_7(r_1, -r_2, -r_3, r_4)$$

play the same part that the inscribed sphere centre I plays, for if we take the sphere centre I_5 and radius $r_5=3V/(A+B-C-D)$ we have

$$\frac{2}{r_5}+\frac{1}{r_1}+\frac{1}{r_2}-\frac{1}{r_3}-\frac{1}{r_4}=0,$$

and I_5 is the centroid of masses

$$1/r_1 \text{ at } I_1,\quad 1/r_2 \text{ at } I_2,\quad -1/r_3 \text{ at } I_3 \text{ and } -1/r_4 \text{ at } I_4.$$

If spheres are described with centres I_1, I_2, I_3, I_4, I_5 and radii $(\rho-r_1)$, $(\rho-r_2)$, $(\rho-r_3)$, $(\rho-r_4)$, $(\rho+r_5)$ respectively, then calling these spheres S_1, S_2, S_3, S_4, S_5 it will be found that the radical

planes of S_5, S_1; S_5, S_2; S_5, S_3; S_5, S_4 meet in a point. This point will lie on S_5 if

$$y^2=\rho_1^2-I_5O_1^2+(\rho+r_5)^2,$$

where $y^2=\rho_1^2-\frac{3}{2}\Sigma(M_1V_1/A)+\frac{1}{16}(\Sigma M_1^2-2\Sigma M_1M_2\cos AB)$

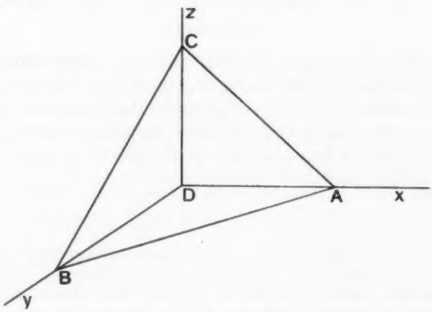
and $$\rho=\frac{1}{4}\left\{\frac{2T_5^2}{r_5}+\frac{T_1^2}{r_1}+\frac{T_2^2}{r_2}-\frac{T_3^2}{r_3}-\frac{T_4^2}{r_4}\right\},$$

and M_1, M_2, ... are functions similar to L_1, L_2, ... That is to say that if

$$y^2=\rho_1^2-I_5O_1^2+(\rho+r_5)^2$$

a sphere can be drawn to touch internally the inscribed spheres centres I_1, I_2, I_3, I_4, and also to touch externally the inscribed sphere centre I_5. If the radius is negative, the words "externally" and "internally" must be interchanged.

6. *To show that in the case of a tetrahedron ABCD in which the edges DA, DB, DC are each equal to a and the angles ADB, ADC, BDC are right angles, a sphere can be found which touches the five inscribed spheres.*



In the tetrahedron $A=B=C=\frac{1}{2}a^2$, $D=\frac{1}{2}a^2\sqrt{3}$, $V=\frac{1}{6}a^3$.

$$r=\tfrac{1}{6}a(3-\sqrt{3})\,;\quad r_1=r_2=r_3=\tfrac{1}{2}a(\sqrt{3}-1)\,;\quad r_4=\tfrac{1}{6}a(3+\sqrt{3}).$$

$$T^2=\tfrac{1}{12}a^2(2\sqrt{3}-1)\,;\quad T_1^2=T_2^2=T_3^2=\tfrac{1}{4}a^2(13-6\sqrt{3})\,;$$

$$T_4^2=-\tfrac{1}{12}a^2(2\sqrt{3}+1).$$

The radius ρ of the sphere which touches the five inscribed spheres is given by

$$\rho=\frac{1}{12}\left[\sum\frac{T_1^2}{r_1}-\frac{T^2}{r}\right].$$

Thus $$\rho=\tfrac{1}{3}a(\sqrt{3}-1).$$

$$L_1=[T_1^2-T^2-2\rho(r+r_1)]=\tfrac{1}{9}a(11\sqrt{3}-15)$$

and $$L_2=L_3=L_1,\quad L_4=\tfrac{1}{3}a(15-11\sqrt{3}).$$

Thus $$L_1+L_2+L_3+L_4=0.$$

If we take Cartesian coordinates, DA, DB, DC being the axes of x, y, z, and if (x, y, z) are the coordinates of the centre of the sphere

touching the five inscribed spheres, $(\alpha, \beta, \gamma, \delta)$ being the quadriplanar coordinates of the same point, then

$$x = \alpha = 3V_1/A + \tfrac{1}{4}[-L_1 + L_2 \cos AB + L_3 \cos AC + L_4 \cos AD]$$
$$= \tfrac{1}{2}a + \tfrac{1}{4}a\{-\tfrac{1}{9}(11\sqrt{3} - 15) + (15 - 11\sqrt{3})/\sqrt{3}\} = \tfrac{1}{9}a\sqrt{3}$$

and $$y = z = \tfrac{1}{9}a\sqrt{3}.$$

From these it will be found that, if Ω is the centre of the sphere,

$$I\Omega = \rho - r,$$
$$I_1\Omega = \rho + r_1, \quad I_2\Omega = \rho + r_2, \quad I_3\Omega = \rho + r_3, \quad I_4\Omega = \rho + r_4,$$

and it will also be found that

$$x^2 = \rho_1^2 - IO_1^2 + (\rho - r)^2$$

where $$x^2 \equiv \rho_1^2 - \tfrac{3}{2}\Sigma(L_1V_1/A) + \tfrac{1}{16}(\Sigma L_1^2 - 2\Sigma L_1L_2 \cos AB).$$

7. The theorem can be stated thus: in any tetrahedron, Ω and ρ being as defined above,

$$I\Omega^2 - (\rho - r)^2 = I_1\Omega^2 - (\rho + r_1)^2 = \ldots = I_4\Omega^2 - (\rho + r_4)^2$$
$$= k^2, \text{ say};$$

that is, the sphere centre Ω and radius k cuts orthogonally the spheres S, $S_1 \ldots S_4$. When $k = 0$, as sometimes happens, the sphere centre Ω and radius ρ touches internally the inscribed sphere centre I and touches externally the escribed spheres centres $I_1, \ldots I_4$. A similar result holds when the inscribed sphere centre I is replaced by any one of the inscribed spheres centres I_5, I_6, I_7.

R. T. R.

1128. The notion of unity that a child picks up either from general conversation or from school teaching is extremely bewildering and paralysing to his mind. He gets the notion connected in his mind with that of zero; instead of with that of a mere datum; he thinks of our arbitrary "one" as an absolute "one"; and hence it is that to not a few men, and, till within the last few years, to the great majority of women, a fraction remained unthinkable. A certain step, of course, is made—a certain hint, at least, is given to any one who thinks—when he hears that our Fahrenheit zero is not the zero of heat.—R. L. Stevenson, *Memories and Portraits*, Tusitala edition, p. 181. [Per Mr. C. E. Kemp.]

1129. The first proposition of Euclid, for instance, is entirely similar to the "twice two is four"; or is a case of what I have already called the "juggle of x and z". You first make the sides equal; and then you remind the reader that you have so made them. When you come to such a proposition, however, as the forty-seventh, there is an appearance of something higher and more abstruse than this logical game of handy-dandy; but where the difference exactly lies and whether it is anything more than a mere appearance, I cannot at present see.—R. L. Stevenson, *Memories and Portraits*, Tusitala edition, p. 187. [Per Mr. C. E. Kemp.]

1130. A few feet below them . . . lay the unmistakable traces of the impact of a body falling from rest with an acceleration due to something more than the force of gravity.—Ian Hay, *A Man's Man*. [Per Mr. D. F. Ferguson.]

A SIMPLE GEOMETRICAL DEVICE, AND SOME OF ITS APPLICATIONS.

BY W. J. DOBBS.

I. Let O, A, B, C be any four points not necessarily in one plane. Let H and K divide OA and BC proportionally, while M and N divide OB and AC proportionally. Then we shall prove that HK and MN meet at T such that OA, MN, BC are proportionally divided at H, T, K, while OB, HK, AC are proportionally divided at M, T, N, the geometrical relations being conveniently, and diagrammatically, expressed thus :

$$\begin{array}{cccccc} & B & — & K & — & C \\ 1-\mu & | & & | & & | \\ & M & — & T & — & N \\ \mu & | & & | & & | \\ & O & — & H & — & A \\ & & \lambda & & 1-\lambda & \end{array}$$

Since H and K divide OA and BC proportionally, we may regard O and B, H and K, A and C as successive simultaneous positions of two particles travelling uniformly along OA and BC. Their mass-centre also travels uniformly along a straight line, and as the mass-ratio may be what we please, the proposition is proved.

II. *Another proof.* Take any plane through O as a plane of reference, and let the distance of any point from this plane *estimated in any assigned direction* be denoted by the corresponding small letter. Then, if $OH/OA=\lambda=BK/BC$ and $OM/OB=\mu=AN/AC$, we have $h=\lambda a$ and $k=\lambda c+(1-\lambda)b$. If T divides HK in the ratio $\mu:1-\mu$, we have

$$t=\mu k+(1-\mu)h=\lambda(1-\mu)a+(1-\lambda)\mu b+\lambda\mu c.$$

This result is unaltered by the interchange of a with b and λ with μ; also it is true for every plane of reference through O. Hence the same point T also divides MN in the ratio $\lambda:1-\lambda$.

III. When $\lambda=\frac{1}{2}$, so that H and K are the middle points of OA and BC, not only is $BK:KC=OH:HA$, but also $CK:KB=OH:HA$. Hence for the same four points O, A, B, C, and for all values of μ and ν, we have simultaneously the two configurations :

$$\begin{array}{cccccc} & B & — & K & — & C \\ 1-\mu & | & & | & & | \\ & M & — & T & — & N \\ \mu & | & & | & & | \\ & O & — & H & — & A \\ & & \frac{1}{2} & & \frac{1}{2} & \end{array} \qquad \text{and} \qquad \begin{array}{cccccc} & C & — & K & — & B \\ 1-\nu & | & & | & & | \\ & U & — & W & — & V \\ \nu & | & & | & & | \\ & O & — & H & — & A \\ & & \frac{1}{2} & & \frac{1}{2} & \end{array}$$

If, in addition, we make $\nu=\mu$ the point W coincides with T. Thus MN and UV now bisect each other at T. Hence, *if we stratify any tetrahedron $OABC$ by plane sections parallel to two opposite edges, each section is a parallelogram and the join of the middle points of these two opposite edges passes through the centre of every such parallelogram.*

If we make $\lambda=\mu=\nu=\frac{1}{2}$, so that H and K, M and N, U and V are the middle points of OA and BC, OB and AC, OC and AB respectively, then HK, MN, UV bisect each other at T. In this case $t=\frac{1}{4}(a+b+c)=\frac{3}{4}g$, where G is the centroid of ABC. Thus *T divides OG in the ratio* 3 : 1.

IV. Now let O, A, B, C be *not* in one plane, and consider again the original configuration :

$$\begin{array}{cccccc} & B & — & K & — & C \\ 1-\mu & | & & | & & | \\ & M & — & T & — & N \\ \mu & | & & | & & | \\ & O & — & H & — & A \\ & & \lambda & & 1-\lambda & \end{array}$$

Take OA, OB, OC as axes of x, y, z and let $OA=a$, $OB=b$, $OC=c$. Then the coordinates of T are :

$$x=\lambda(1-\mu)a,$$
$$y=(1-\lambda)\mu b,$$
$$z=\lambda\mu c.$$

Eliminating λ and μ, we have $(x/a+z/c)(y/b+z/c)=z/c$.

Now for varying values of λ and μ the point T may occupy any position in any line of the figure. Hence the various positions of HK (including OB and AC) constitute one system of generating lines of the above surface no two of which are in one plane, while the different positions of MN (including OA and BC) constitute another system of such generating lines of the same surface, and every member of either system meets all members of the other. *If a variable member of one system meets three fixed members of the other system at P, Q, R, then the ratio $PQ : PR$ is constant.*

V. As HK and MN meet at T, the points H, K, M, N are in one plane which is fully determined by any three of them. Hence *any plane which cuts two opposite edges OA and BC of a tetrahedron $OABC$ proportionally cuts also the two opposite edges OB and AC proportionally.*

In particular, if H and K are the middle points of OA and BC, the same plane which passes through H and K and divides OB at M and AC at N each in the ratio $\mu : 1-\mu$ will also divide OC at U and AB at V each in the ratio $\nu : 1-\nu$, where μ and ν are no longer independent. Also HK bisects MN at T and UV at W.

Now this plane meets the face ABC in a straight line: hence K, N, V are collinear. Similarly NHU, KMU and MHV are

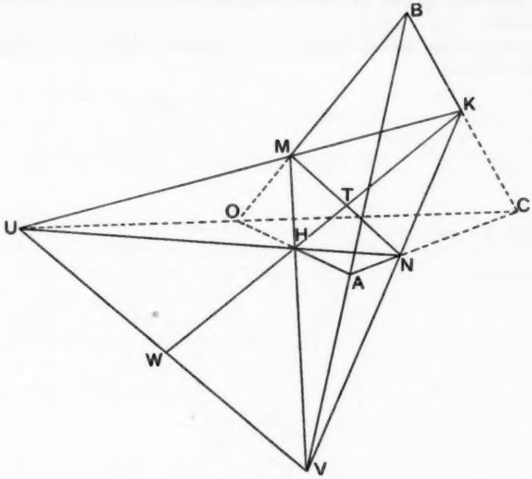


FIG. 1.

straight lines. And as MN and UV are both bisected by HK, it follows that *MN and UV are parallel.*

But HK is divided harmonically at T and W, so that

$$UO/UC = VA/VB = OM/MB = AN/NC.$$

Also $$\mu/(1-\mu)+\nu/(1-\nu)=0,$$
whence $$\nu=\mu/(2\mu-1).$$

We see that, in general, a plane which bisects two opposite edges of a tetrahedron divides another pair of opposite edges internally and the third pair externally.

The limitation that O, A, B, C should *not* be in one plane may now be removed, since parallel projection of the figure upon a plane leaves parallelism, concurrency and the ratio of the segments of a straight line unaffected. The properties of the figure therefore remain unchanged.

VI. Now let $OABC$ be any tetrahedron, H and K the middle points of the two opposite edges OA and BC, and M any point in OB between O and B. Then the plane MHK will cut AC at N between A and C, and HK bisects MN (Fig. 2).

Thus M and N are at equal perpendicular distances on opposite sides of the plane BHC.

Also the triangles BHK and CHK are equal in area: hence

$$\text{volume } MBHK = \text{volume } NCHK.$$

Now volume $BHAC$ is half the volume of the original tetrahedron. Removing the portion $NCHK$ and adding the equal portion $MBHK$, we see that the plane $MHNK$ bisects the volume of the original tetrahedron.

Thus *any plane which bisects two opposite edges of a tetrahedron bisects its volume.*

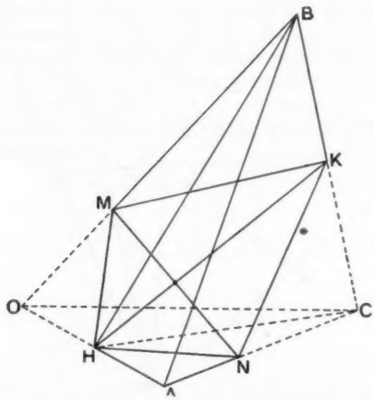


FIG. 2.

The following proof of this theorem is due to Dr. G. T. Bennett: Stratify the tetrahedron by plane sections parallel to two opposite edges. Each section is a parallelogram, and the join of the middle points of these two edges passes through the centre of each parallelogram. Any plane containing this join therefore bisects the area of every parallelogram and consequently bisects the volume of the tetrahedron.

VII. Now let the four points O, A, B, C be in one plane. Ignoring the cases in which OA is parallel to BC and/or OB to AC (which merely lead to a system of parallel lines crossed by a system of parallel or concurrent lines), we assume that OA meets BC at E and that H and K do not simultaneously arrive at E. Keeping μ constant (not zero or unity), MN will not pass through E but will meet OA at Q and BC at R. Allowing λ to vary, H will travel along OA, T along MN, K along BC. When H is at Q, T remaining always collinear with H and K, it is clear that K is at R and T at P dividing QR in the ratio $\mu : 1-\mu$. Thus we see that for every position of MN there is one position QR of HK in line with MN. Hence HK envelops the same curve as MN; there are no longer two systems of intersecting lines, but one system of tangents to the same curve.

Also, as T tends towards the position P when HK moves into line with MN, we see that P is the point of contact of MQ with the curve which it envelops.

Similarly we can determine the points of contact with the same curve of OA, OB, AC, BC, HK.

Notice that $OQ/OA = MP/MN = BR/BC$.

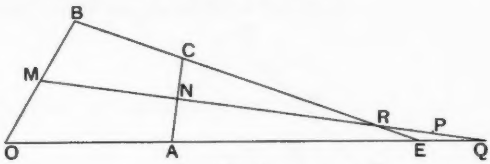


FIG. 3.

VIII. It is, however, possible to simplify greatly the investigation without further loss of generality. Let H and K travel uniformly along two intersecting lines and not arrive simultaneously at their point of intersection, which we will now denote by O. Assume that H is the first to arrive at O when K is at B moving towards O, and let H move from O to A while K moves from B to O. In this way we have made the point C coincide with O, and our configuration becomes :

$$\begin{array}{cccccc} & B & \text{——} & K & \text{——} & O \\ 1-\mu & | & & | & & | \\ & M & \text{——} & T & \text{——} & N \\ \mu & | & & | & & | \\ & O & \text{——} & H & \text{——} & A \\ & & \lambda & & 1-\lambda & \end{array}$$

It is now clear that NM and HK are interchangeable, λ with $1-\mu$ and μ with $1-\lambda$.

As $\mu \to 1-\lambda$ the configuration tends towards :

$$\begin{array}{cccccc} & B & \text{——} & K & \text{——} & O \\ \lambda & | & & | & & | \\ & K & \text{——} & P & \text{——} & H \\ 1-\lambda & | & & | & & | \\ & O & \text{——} & H & \text{——} & A \\ & & \lambda & & 1-\lambda & \end{array}$$

Thus P is the point of contact of HK with its envelope, where

$$BK/KO = KP/PH = OH/HA \text{ (Fig. 4).}$$

IX. Now describe two circles, one to touch OB at O and pass through A, the other to touch OA at O and pass through B, and let these circles meet again at S (Fig. 5).

Then $\qquad \angle SOA = \angle SBO \quad$ and $\quad \angle SAO = \angle SOB$.

Thus the triangles SOA and SBO are similar, and so

$$SO/SB=OA/BO=OH/BK.$$

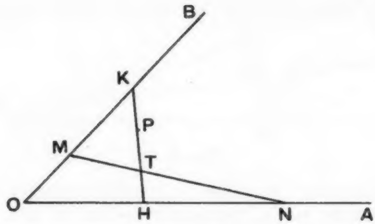


Fig. 4.

And, as $\angle SOH=\angle SBK$, the triangles SOH and SBK are also similar.

Hence $$SH/SK=OH/BK=OA/BO.$$

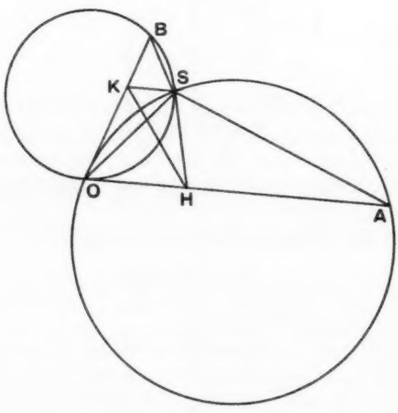


Fig. 5.

Hence *the distances of H and K from the fixed point S are in a constant ratio* (*that of their speeds*)*;* also *SH and SK have equal angular velocities in the same sense.*

As $\angle OHS=\angle BKS$, the points O, H, S, K are concyclic. Hence the foot of the perpendicular from S upon HK is situated in the fixed line joining the feet of the perpendiculars from S upon OA and OB. Hence *HK envelops a parabola of focus S.*

X. Take OA and OB as axes of x and y, and let $OA=a$ and $OB=b$. Then the coordinates of T are $[\lambda(1-\mu)a, (1-\lambda)\mu b]$, and the coordinates of P are $[\lambda^2 a, (1-\lambda)^2 b]$.

The equation of the parabola is therefore

$$\pm\sqrt{\frac{x}{a}}\pm\sqrt{\frac{y}{b}}=1,$$

and the equation of the tangent HK is

$$\frac{x}{\lambda a}+\frac{y}{(1-\lambda)b}=1,$$

i.e. $$\frac{x}{\alpha}+\frac{y}{\beta}=1 \text{ subject to } \frac{\alpha}{a}+\frac{\beta}{b}=1.$$

Other properties of the parabola also become obvious. For instance: *If a variable tangent to a parabola meets three fixed tangents at P, Q, R, then the ratio PQ : PR is constant.* W. J. D.

1131. Motorist.

Mr. J. F. Timms, of Brackley, has kindly drawn my attention to a real long-distance motorist. He has sent me a cutting which says:

"Joseph Patrick Abbott, of Olympia-street, Burnley, Lancs., was stated at Clerkenwell Police Court to have driven a motor-car 5,500,000 miles during eighteen years. He was fined £3 and £2 3s. costs for having driven without due care and attention, and had his licence endorsed. He collided with another car on road crossings where there are traffic lights."

Mr. Abbott has been having a hard time these last eighteen years.

By the aid of three professors of mathematics, four slide rules and an adding machine, I have worked out this motorist must have covered over 800 miles a day.

Assume that he gets eight hours' sleep a day, then Mr. Abbott travels at a steady speed of 52 miles per hour without stopping for meals.

An incidental expense at 30 miles to the gallon is just over £500 per annum for petrol.

Now that they've been wasting his time at a police court he'll have to step on it to maintain his average.

Let me know when you are coming my way, Mr. Abbott, and I'll give you a cheer!—*Daily Mirror*, December 23, 1936. [Per Mr. R. O. Street.]

1132. I became friendly with the old chap who had been a schoolmaster and was a grand mathematician; he taught me the Calculi and nearly set me up for life in gaol extras.—W. Macartney, *Walls have Mouths*, p. 322. [Per Mr. C. E. Kemp.]

1133. . . . it may be a mark of Shaw's genius that he can divide the indivisible and thus square the crooked circle.—Frank Harris, *Bernard Shaw*, p. 308. [Per Mr. C. E. Kemp.]

1134. A panoramic show like the present is a series of historical "ordinates" (to use a term in Geometry): the subject is familiar to all; and foreknowledge is assumed to fill in the curves required to combine the whole gaunt framework into an artistic unity.—Thomas Hardy, Preface to *The Dynasts*. [Per Mr. C. E. Kemp.]

ON THE REPRESENTATION OF CIRCLES BY MEANS OF POINTS IN SPACE OF THREE DIMENSIONS.

By D. Pedoe.

The idea of the representation of a linear system of plane curves by means of a flat space is of fundamental importance in algebraic geometry. In this paper some details of the representation of circles are worked out, and an attempt is made to give a three-dimensional picture of certain parts of circle geometry. The theorems given in *italics* are thought to be new, although in a subject with such an enormous literature it might almost be said that no theorem can be a new one.

Let the equation of a circle C in the (x, y) plane be

$$C \equiv x^2 + y^2 - 2\xi x - 2\eta y + \zeta = 0.$$

We represent this circle by the point P with coordinates (ξ, η, ζ) in Euclidean space of three dimensions, which we refer to subsequently as S_3. If our coordinate axes are suitably chosen, the projection of P on to the (x, y) plane will be the centre of the circle C.

This representation evidently sets up a one-to-one correspondence between the circles of the plane and the points of S_3. Circles of zero radius are represented by points on the paraboloid of revolution

$$\Omega \equiv \xi^2 + \eta^2 - \zeta = 0.$$

Circles with real radius are represented by points outside Ω, and circles with imaginary radius by points inside Ω. We call these latter circles " imaginary circles ".

The circles of the coaxal system

$$x^2 + y^2 - 2\xi x - 2\eta y + \zeta + \lambda[x^2 + y^2 - 2\xi' x - 2\eta' y + \zeta'] = 0$$

are represented by the points

$$\left(\frac{\xi + \lambda\xi'}{1+\lambda}, \frac{\eta + \lambda\eta'}{1+\lambda}, \frac{\zeta + \lambda\zeta'}{1+\lambda}\right),$$

i.e. by the points on the line joining P to P'. If this line meets Ω in real points, the coaxal system contains circles of zero radius, *i.e.* the intersections of the line and Ω give the limiting points of the coaxal system.

A coaxal system of circles determines a pair of real or conjugate imaginary points in the plane, namely the common intersections of the circles. Conversely it is determined by such a pair of points. Hence our representation can be regarded as also setting up a correspondence between the lines of S_3 and the pairs of real or conjugate imaginary points of the plane. Two incident lines of S_3 correspond to two coaxal systems with a common circle, and therefore to four concyclic points. Remembering the theorem that four general lines of S_3 have just two transversals, we obtain the theorem :

Given four pairs of points PP', QQ', RR', SS' in a plane, the problem of finding another pair XY, such that $PP'XY$, $QQ'XY$, $RR'XY$ and $SS'XY$ are respectively concyclic admits of two solutions.

These need not be real, and of course, if there are more than two there must be an infinity.

Other theorems may be obtained from other incidence theorems for S_3. For example, the theorem that through a point one transversal can be drawn to two skew lines leads to an interesting theorem. We now turn to properties of circles which bring in Ω more directly.

The condition that two circles C_1, C_2 should intersect at an angle α or $\pi-\alpha$ is easily found to be

$$(2\xi_1\xi_2+2\eta_1\eta_2-\zeta_1-\zeta_2)^2=4(\xi_1{}^2+\eta_1{}^2-\zeta_1)(\xi_2{}^2+\eta_2{}^2-\zeta_2)\cos^2\alpha.$$

We first examine the special cases $\alpha=0$, $\alpha=\pi/2$. When $\alpha=0$, the condition that C_1, C_2 should touch is seen to be just the condition that the line P_1P_2 should touch Ω. In other words, the necessary and sufficient condition that two circles should touch is that the limiting points of the coaxal system they determine should coincide.

When $\alpha=\frac{1}{2}\pi$, we see that P_1, P_2 are conjugate with regard to Ω. Two lines which are such that every point of one is conjugate to every point of the other, *i.e.* two lines *polar* with regard to Ω represent two conjugate systems of coaxal circles. A line of S_3 projects on the (x, y) plane into the line of centres of the coaxal system it represents. Therefore polar lines project into lines at right angles. The vertices of a tetrahedron self-conjugate with regard to Ω project into a triangle and its orthocentre. The plane theorem, that of four mutually orthogonal circles, one must be imaginary, corresponds to the theorem that one vertex of a self-conjugate tetrahedron must lie inside Ω.

The points representing circles which cut a fixed circle C' at an angle α or $\pi-\alpha$ fill the quadric

$$4(\xi^2+\eta^2-\zeta)(\xi'^2+\eta'^2-\zeta')\cos^2\alpha=(2\xi\xi'+2\eta\eta'-\zeta-\zeta')^2.$$

This quadric has ring contact with Ω along its intersection with the plane

$$2\xi\xi'+2\eta\eta'-\zeta-\zeta'=0.$$

The pole of this plane is the point (ξ', η', ζ'), and the projection of the conic in which the plane meets Ω is

$$x^2+y^2-2\xi'x-2\eta'y+\zeta'=0,$$

i.e. the circle of which (ξ', η', ζ') is the representative point. The conic on Ω represents the circle when the circle is considered as a locus of point-circles, *i.e.* of points. Similarly any curve on Ω is the representative of a curve in the plane considered as a locus of points, and the curve in the plane is obtained by orthogonal projection of the curve on Ω.

To investigate the circles cutting three given circles $C_r\,(r=1, 2, 3)$ at angles α or $\pi-\alpha$, we must examine the common intersections of the three quadrics

$$4(\xi_r{}^2+\eta_r{}^2-\zeta_r)(\xi^2+\eta^2-\zeta)\cos^2\alpha=(2\xi\xi_r+2\eta\eta_r-\zeta-\zeta_r)^2,\quad (r=1, 2, 3)$$

For the sake of brevity we write these equations as

$$4k_r(\xi^2+\eta^2-\zeta)\cos^2\alpha = X_r^2, \qquad (r=1,2,3)$$

and instead of writing "α or $\pi-\alpha$" we write "$\pm\alpha$". Three quadrics with no special relation to each other have eight intersections, and there are therefore eight circles cutting three given circles at angles $\pm\alpha$. It is interesting to see what happens as we vary α.

The locus of the eight points is found by eliminating $\cos^2\alpha$ from the three equations, when we have

$$\frac{X_1^2}{k_1}=\frac{X_2^2}{k_2}=\frac{X_3^2}{k_3}.$$

This system is equivalent to the system

$$\begin{cases}\sqrt{k_2}X_1 \pm \sqrt{k_1}X_2 = 0,\\ \sqrt{k_3}X_2 \pm \sqrt{k_2}X_3 = 0,\end{cases}$$

giving four lines passing through the point

$$X_1 = X_2 = X_3 = 0.$$

This point represents the circle orthogonal to the three given circles. We have proved then that "the circles intersecting three fixed circles in equal or supplementary (non-specified) angles form four coaxal systems with one common circle, the circle orthogonal to the three circles". Reversing the solution of our final set of equations, we see that the four coaxal systems correspond to circles intersecting the given circles in the angles:

$$\left.\begin{matrix}\alpha, & \alpha, & \alpha\\ -\alpha, & -\alpha, & -\alpha\end{matrix}\right\} \quad \left.\begin{matrix}\alpha, & \alpha, & -\alpha\\ -\alpha, & -\alpha, & \alpha\end{matrix}\right\} \quad \left.\begin{matrix}\alpha, & -\alpha, & -\alpha\\ -\alpha, & \alpha, & \alpha\end{matrix}\right\} \quad \left.\begin{matrix}\alpha, & -\alpha, & \alpha\\ -\alpha, & \alpha, & -\alpha\end{matrix}\right\},$$

each bracket representing two circles in the same coaxal system.

Coaxal systems are the simplest kind of algebraic systems of circles. We naturally define an algebraic system of circles as one represented by an algebraic curve in S_3. After the straight line, it is natural to consider the conic in S_3. A system of circles represented by a conic in S_3 is called a "conic system". All circles of a conic system are orthogonal to a fixed circle, represented in S_3 by the pole (with regard to Ω) of the plane of the conic, and the centres of circles of a conic system lie on a conic, the projection of the conic in S_3. It is easily seen that these two properties suffice to define a conic system uniquely. After these remarks we turn to consider the *envelope* of an algebraic system of circles.

Let $\mathcal{C}$ be a curve in S_3 representing the system. The envelope of the system is given by the locus of ultimate intersections of "consecutive" circles of the system. If P is a point on $\mathcal{C}$, the tangent at P to $\mathcal{C}$ represents a coaxal system of circles whose common points are two of such ultimate intersections. The line polar to this tangent (with regard to Ω) gives the conjugate coaxal system, whose limiting

points are these ultimate intersections. The *actual* limiting points in the plane are the projections of the points in which this polar line meets Ω. Therefore, *to obtain the envelope of the system of circles represented by a curve $\mathcal{C}$ in S_3, we find the curve in which lines polar to the tangents of $\mathcal{C}$ (with respect to Ω) meet Ω, and project this curve orthogonally on to the (x, y) plane.*

The envelope of a conic system is now easily obtained. The lines polar to the tangents to a conic in S_3 generate a quadric cone. This meets Ω in a quartic curve. Suppose that $(x, y, z, 1)^2=0$ is the equation of any quadric cutting this quartic curve on Ω. Then $(x, y, x^2+y^2, 1)^2=0$ is the equation of the projection of the curve, *i.e.* the projection is a bicircular quartic. The envelope of a conic system of circles is therefore a bicircular quartic. Any bicircular quartic can be obtained in this way, and so the properties of the curve can be developed. For example, through the quartic curve on Ω there passes a pencil of quadrics, and the pencil contains four cones. The polar reciprocals of these four cones (with regard to Ω) give four conics. Hence a bicircular quartic can be generated as the envelope of four systems of circles. Again, since a conic meets Ω in four points, each enveloping system of circles contains four point-circles, and so the bicircular quartic has sixteen foci. Each set of four foci is the intersection of a conic and a circle, the circle being the fixed circle to which all circles of the corresponding conic system are orthogonal. Once the envelope property has been established, the development of the properties of this interesting quartic curve must necessarily proceed on similar lines to the geometric treatment given in ch. 3 of vol. 4 of Professor Baker's *Principles of Geometry*. We therefore proceed with the representation in S_3 of inversion in a circle.

If A, A' are inverse points in a circle C_1, represented by the point P_1 with coordinates (ξ_1, η_1, ζ_1), it is easily seen that C_1 is a member of the coaxal system determined by the point circles A and A'. So that if A and A' be respectively projected up into points P and P' on Ω, P_1, P and P' must be collinear. Hence, to obtain the inverse of a curve in the plane we may project it up on to Ω, join the resulting curve on Ω by a cone of lines to P_1, and project down on to the plane again the further curve in which this cone meets Ω. We can now obtain theorems for Ω from the known theorems of inversion. The theorems that the inverse of a circle is a circle, and that a circle, the circle of inversion and the inverse circle are coaxal become the following : " A quadric cone which meets Ω in one plane section will meet it again in another plane section, and the poles of the plane sections will be collinear with the vertex of the cone." *

The construction in S_3 for the inverse of a curve is an interesting one in that it sometimes gives more than we expect, as we shall show. To avoid the introduction of too many symbols, we shall sometimes use the same symbol for a plane in S_3, the conic it cuts on Ω, and the circle in the plane of (x, y) into which the conic pro-

* See remark at end of paper.

jects. The context will clearly show which entity is symbolised at any moment.

Let a fixed plane cut Ω in a conic D, and let P (a point representing a circle C) vary in a plane E. Then C is always orthogonal to the circle E. We construct the cone $P(D)$, which meets Ω again in a conic D', and we suppose that P' is the pole of D' with regard to Ω. Then P' represents the inverse of the circle D in the circle C. Now as P varies in E, what is the locus of P'? We find it from the following theorem in inversion.

If a circle C is always orthogonal to a fixed circle E, and C' is the inverse of another fixed circle D in C, then the angle of intersection of C' and E is constant, being equal to the angle of intersection of D and E.

Inversion with respect to a point on C gives an immediate proof of this theorem. It follows (referring back to the beginning of this paper) that P' moves on the quadric

$$4(\xi_1^2+\eta_1^2-\zeta_1)(\xi^2+\eta^2-\zeta)\cos^2\alpha=(2\xi\xi_1+2\eta\eta_1-\zeta-\zeta_1)^2,$$

where $\pm\alpha$ is the angle of intersection of D and E, and (ξ_1, η_1, ζ_1) represents the circle E. This is a quadric having ring contact with Ω along E, and (naturally) passing through the point representing D.

We have now set up a correspondence between a plane E, and a quadric which has ring contact with Ω along the intersection of E and Ω, by means of an intermediary conic D on Ω. Suppose we now take three planes F, G, and H which intersect in a point P, and, as above, let us construct the three corresponding quadrics. These three quadrics will intersect in seven points other than the pole of D, and this at first sight is puzzling. We had expected to obtain only the point which represents the inverse of the circle D in the circle represented by the point P. Our construction evidently does more than this. In fact, if the angles of intersection of the circles F, G, H with the circle D be respectively α, β, γ, then the intersections of the three quadrics represent the circles which intersect the circles F, G, and H at angles $\pm\alpha$, $\pm\beta$, $\pm\gamma$. It is easy to prove that, besides D, the inverse of D in the circle represented by P satisfies these conditions, and D the other six circles must be avoided if we are only interested in inverses.

If we consider only two planes F and G, and their corresponding quadrics, the preceding work is of service. Two planes will intersect in a line, and we suppose this line to meet Ω in two real points, X and Y. The two quadrics will touch Ω at X and Y, and will therefore intersect in two conics which each touch Ω at X and Y. The question we ask is, "What locus represents the inverse of a fixed circle D in the circles of a coaxal system represented by the line XY?" The previous paragraph enables us to say that it is that one of the two conics touching Ω which passes through the point representing D.* The presence of the other conic can be explained as above.

* This point is on the locus, for one circle of the coaxal system is orthogonal to D, and the inverse of D in this circle is D itself.

The envelope of the conic series of circles we have constructed can be found as described in an earlier part of the paper. Denote the conic by $\mathcal{C}$, and the conic in which its plane cuts Ω by H, and let Q be the pole of H with respect to Ω. Then the polar lines of the tangents to $\mathcal{C}$ all pass through Q, and meet the plane of $\mathcal{C}$ in the polar reciprocal of $\mathcal{C}$ with respect to H.

Since $\mathcal{C}$ and H have double contact at X, Y, the polar reciprocal likewise has double contact with H at X, Y, and so the envelope of the conic system $\mathcal{C}$ is given by the projection of the curve in which a cone, vertex Q and touching Ω at X and Y, meets Ω. This curve consists of two conics through X, Y, and these project into two circles through the limiting points of the coaxal system represented by XY. Hence:

*If we invert a fixed circle in the circles of a non-intersecting coaxal system, the resulting system of circles is orthogonal to a fixed circle, the centres lie on a conic which has double contact with this circle at the limiting points of the coaxal system, and the envelope of the system consists of two circles of the conjugate coaxal system.**

We conclude with this application of the methods outlined, and remark that these methods appear to be capable of many further applications. It may also be remarked that the projective theorems proved true for Ω must also be true for any non-degenerate quadric, and so a simple approach to three-dimensional geometry is indicated.

D. P.

1135. Hence a degree of defect that may interfere with the study of algebra need not seriously handicap the individual in harnessing horses or cleaning motor-vehicles.—H. L. Hollingworth, *Abnormal Psychology*, p. 188. [Per Mr. J. B. Bretherton.]

1136. Another notable feature is the fact that the seadrome is not anchored in position. Air currents agitate water to a depth of about 60 ft., at which point pressure starts and increases with the depth until there would be sufficient upward force to sustain the structure of the seadrome in position as the centrifugal force of the world keeps the ocean in place.—From a paragraph relating to a picture of a projected floating seadrome; *Illustrated London News*, February 27, 1937, p. 355. [Per Mr. A. R. Pargeter.]

1137. "... Come and sit down here beside me and tell me what you've been doing since I saw you last. Two whole days."

"No; one whole day."

"But it's forty-eight hours."

"Ah, that may be; but only Friday was whole. I saw you on Thursday and now I'm seeing you on Saturday. Thus one *whole* day. Extraordinary how few women have any head for mathematics."—P. C. Wren, *Explosion*. [Per Mr. J. P. McCarthy.]

* The reader may verify, that if the fixed circle be taken as

$$x^2+y^2+2gx+2fy+d=0,$$

and the coaxal system as $x^2+y^2+2\lambda x+c^2=0$, the envelope is

$$(x^2+y^2+2\lambda y-c^2)(x^2+y^2+2\mu y-c^2)=0,$$

where λ, μ are the roots of

$$4t^2(d-g^2)-4ft(d-c^2)+(c^2+d)^2-4c^2(f^2+g^2)=0.$$

FRACTIONAL CALCULUS.

By W. Fabian.

1. We continue here our work on fractional integration and differentiation of functions of a real variable which we began in a previous paper.* All quantities in the present paper are real.

We define a λth integral, or a $(-\lambda)$th differential coefficient, of $f(x)$ over an interval (a, x) by *

$$D^{-\lambda}(l_a)f(x)=\frac{D^{\gamma}}{\Gamma(\lambda+\gamma)}\int_a^x (x-t)^{\lambda+\gamma-1}f(t)\,dt,$$

where D stands for $\frac{d}{dx}$, and γ is the least integer greater than or equal to zero such that $\lambda+\gamma>0$; $f(x)$ is bounded on the path of integration, and is continuous on this path except possibly for a finite number of ordinary discontinuities.

In the present paper the principal value of $(x-t)^{\lambda+\gamma-1}$ in the definition of $D^{-\lambda}(l_a)f(x)$ is to be taken, and λ and a are to be taken as finite. When no ambiguity can arise, we shall write $f_\lambda(x)$ for $D^{-\lambda}(l_a)f(x)$.

2. *Theorem* 1. *Let $f(x)$ be bounded and either positive or negative throughout (a, b), the upper and lower bounds of $|f(x)|$ being M and m respectively. Let*

$$\frac{M(c-a)^{\lambda}}{\{(\lambda-1)p+1\}^{\frac{1}{p}}}<\frac{m}{\lambda}(d-a)^{\lambda},$$

where $\quad p>1, \quad \lambda>0, \quad (\lambda-1)p>-1, \quad a\leqslant c<d\leqslant b.$

Then $f_\lambda(x)$ at every point in (d, b) is greater or less than $f_\lambda(x)$ at every point in (a, c), according as $f(x)$ is positive or negative in (a, b).

Proof. Let x' be any point in (a, c), and x'' any point in (d, b).

Since
$$\frac{M(c-a)^{\lambda}}{\{(\lambda-1)p+1\}^{\frac{1}{p}}}<\frac{m}{\lambda}(d-a)^{\lambda},$$

therefore
$$\frac{M(x'-a)^{\lambda}}{\{(\lambda-1)p+1\}^{\frac{1}{p}}}<\frac{m}{\lambda}(x''-a)^{\lambda}. \qquad \text{.....................(1)}$$

By Hölder's inequality,

$$\left|\int_a^{x'}(x'-t)^{\lambda-1}f(t)\,dt\right|$$

$$\leqslant\left\{\int_a^{x'}|f(t)|^{\frac{p}{p-1}}dt\right\}^{1-\frac{1}{p}}\left\{\int_a^{x'}(x'-t)^{(\lambda-1)p}dt\right\}^{\frac{1}{p}}$$

* Fabian, *Math. Gazette*, vol. 20 (1936), pp. 88-92. In our other previous papers on the Fractional Calculus the complex variable was used.

$$\leqslant M\left(\int_a^{x'} dt\right)^{1-\frac{1}{p}} \left\{\int_a^{x'} (x'-t)^{(\lambda-1)p}\, dt\right\}^{\frac{1}{p}}$$

$$=\frac{M(x'-a)^{\lambda}}{\{(\lambda-1)p+1\}^{\frac{1}{p}}}. \qquad \text{.........(2)}$$

Also $$\left|\int_a^{x''} (x''-t)^{\lambda-1} f(t)\, dt\right| \geqslant \frac{m}{\lambda}(x''-a)^{\lambda}. \qquad \text{.........(3)}$$

By (1), (2) and (3),

$$\left|\int_a^{x'} (x'-t)^{\lambda-1} f(t)\, dt\right| < \left|\int_a^{x''} (x''-t)^{\lambda-1} f(t)\, dt\right|.$$

Therefore $\int_a^{x''} (x''-t)^{\lambda-1} f(t)\, dt$ is greater or less than

$$\int_a^{x'} (x'-t)^{\lambda-1} f(t)\, dt,$$

according as $f(t)$ is positive or negative in (a, b).

Hence the theorem.

From the theorem just proved we can deduce the following theorem :

Theorem 2. *Let $f(x)$ and its first γ non-fractional differential coefficients be finite and continuous in (a, b), where $f(x)$ is such that $f^{(\gamma)}(x)$ is either positive or negative throughout (a, b).*

If $\lambda \leqslant 0$, let $f(a)=f'(a)=\ldots=f^{(\gamma-1)}(a)=0$. Also let

$$\frac{M(c-a)^{\lambda+\gamma}}{\{(\lambda+\gamma-1)p+1\}^{\frac{1}{p}}} < \frac{m}{\lambda+\gamma}(d-a)^{\lambda+\gamma},$$

where $p>1$, $(\lambda+\gamma-1)p>-1$, $a\leqslant c<d\leqslant b$, and M and m are the upper and lower bounds of $|f^{(\gamma)}(x)|$ in (a, b) respectively.

Then $f_\lambda(x)$ at every point in (d, b) is greater or less than $f_\lambda(x)$ at every point in (a, c), according as $f^{(\gamma)}(x)$ is positive or negative in (a, b).

For, integrating $f_\lambda(x)$ by parts γ times, we get, in (a, b),

$$f_\lambda(x)=f^{(\gamma)}_{\lambda+\gamma}(x).$$

On applying now Theorem 1 to $f^{(\gamma)}_{\lambda+\gamma}(x)$, the conclusion follows.

Theorem 3. *Let $f(x)$ and all its non-fractional differential coefficients be finite and continuous in (a, b).*

Let $f_\lambda(x)$, where λ is a negative fraction, be either positive or negative throughout (a, b). Also let

$$\frac{M(c-a)^{-\lambda}}{\{1-(\lambda+1)p\}^{\frac{1}{p}}} < -\frac{m}{\lambda}(d-a)^{-\lambda},$$

where $p>1$, $(\lambda+1)p<1$, $a<c<d\leqslant b$, and M and m are the upper and lower bounds of $|f_\lambda(x)|$ in (a, b) respectively.

Then $f(x)$ at every point in (d, b) is greater or less than $f(x)$ at every point in (a, c), according as $f_\lambda(x)$ is positive or negative in (a, b).

Proof. Integrating $f_\lambda(x)$ by parts γ times, we get, in (a, b),

$$f_\lambda(x)=\sum_{n=0}^{\gamma-1}\frac{f^{(n)}(a)}{\Gamma(\lambda+n+1)}(x-a)^{\lambda+n}+f^{(\gamma)}_{\lambda+\gamma}(x). \quad \text{.............(1)}$$

Since $f^{(\gamma)}(x)$ is finite and continuous in (a, b), and $\lambda+\gamma>0$, therefore * $f^{(\gamma)}_{\lambda+\gamma}(x)$ is finite and continuous in (a, b).

Since $$\frac{M(c-a)^{-\lambda}}{\{1-(\lambda+1)p\}^{\frac{1}{p}}}<-\frac{m}{\lambda}(d-a)^{-\lambda},$$

therefore M is finite. Therefore, by (1),

$$f(a)=f'(a)=\ldots=f^{(\gamma-1)}(a)=0.$$

Hence, in (a, b), by a previous theorem,†

$$D^\lambda f_\lambda(x)=f(x).$$

Applying now Theorem 1 to $D^\lambda f_\lambda(x)$, we obtain the conclusion.

Theorem 4. Let $f(x)$ and all its non-fractional differential coefficients be finite and continuous in (a, b).

Let $f_{\lambda-\delta}(x)$, where λ is a fraction, be finite and continuous, and either positive or negative throughout (a, b), δ being the least integer greater than or equal to zero such that $\delta-\lambda>0$. Also let

$$\frac{M(c-a)^{\delta-\lambda}}{\{1+(\delta-\lambda-1)p\}^{\frac{1}{p}}}<\frac{m}{\delta-\lambda}(d-a)^{\delta-\lambda},$$

where $p>1$, $(\delta-\lambda-1)p>-1$, $a\leqslant c<d\leqslant b$, and M and m are the upper and lower bounds of $|f_{\lambda-\delta}(x)|$ in (a, b) respectively.

Then $f(x)$ at every point of (d, b) is greater or less than $f(x)$ at every point of (a, c), according as $f_{\lambda-\delta}(x)$ is positive or negative in (a, b).

Proof. If $\lambda>0$, let s be an integer such that $\lambda-s>0$. Then, by the first mean-value theorem, for $\lambda>0$,

$$f_{\lambda-s}(x)=\frac{(x-a)^{\lambda-s}}{\Gamma(\lambda-s+1)}f(\xi),$$

where $a\leqslant\xi\leqslant x\leqslant b$.

Hence, if $\lambda>0$, $f_\lambda(a)=f_{\lambda-1}(a)=\ldots=f_{\lambda-\delta+1}(a)=0$,

and $f_\lambda(x)$, $f_{\lambda-1}(x)$, $\ldots f_{\lambda-\delta+1}(x)$ are finite at all points of (a, b). Since, in (a, b), for $\lambda>0$,

$$\int_a^x f_{\lambda-\delta}(x)\,dx=f_{\lambda-\delta+1}(x)-f_{\lambda-\delta+1}(a)=f_{\lambda-\delta+1}(x),$$

$f_{\lambda-\delta}(x)$ being, by hypothesis, finite and continuous in (a, b), therefore $f_{\lambda-\delta+1}(x)$ is finite and continuous in (a, b). And similarly

* Fabian, *Phil. Mag.* vol. 20 (1935), pp. 781-789.

† Fabian, *Math. Gazette*, vol. 20 (1936), pp. 88-92.

$f_{\lambda-\delta+2}(x)$, $f_{\lambda-\delta+3}(x)$, ... $f_\lambda(x)$ are finite and continuous in (a, b) for positive values of λ. Hence $f_\lambda(x)$ and its first δ non-fractional differential coefficients are finite and continuous in (a, b) for all fractional values of λ.

Integrating $f_\lambda(x)$ by parts γ times, we get, in (a, b), for $\lambda<0$,

$$f_\lambda(x)=\sum_{n=0}^{\gamma-1}\frac{f^n(a)}{\Gamma(\lambda+n+1)}(x-a)^{\lambda+n}+f_{\lambda+\gamma}^{(\gamma)}(x).$$

Hence, since $f_\lambda(x)$ and $f_{\lambda+\gamma}^{(\gamma)}(x)$ are bounded in (a, b), we have, for $\lambda<0$,

$$f(a)=f'(a)=\ldots=f^{(\gamma-1)}(a)=0.$$

Therefore, by a previous theorem,* we have, in (a, b), for all fractional values of λ,

$$D^\lambda f_\lambda(x)=f(x).$$

We can now apply Theorem 2 to $D^\lambda f_\lambda(x)$ to obtain the conclusion. W. F.

1138. Sinus (d'où cosinus) terme de géométrie, est la traduction latine d'un mot arabe qui signifie proprt. " pli de vêtement ".—L'Clédat, *Dictionnaire Étymologique de la Langue Française*, p. 579. [Per Mr. J. B. Bretherton.]

1139. ARITHMETICAL QUESTION.

Gentlemen :

An application was made to me last week for the solution of 99*l*. 19*s*. 11¾*d*. multiplied by the same number, when, in the presence of the person deputed, I gave the work as follows :

$$99l.\ 19s.\ 11\tfrac{3}{4}d.=99l.\ \tfrac{959}{960}$$

$$=\frac{95999}{960}\times\frac{95999}{960}=\frac{9215808001}{921600}.$$

$$9999l.\ 15s.\ 10d.\ \tfrac{1}{960}\text{ of a farthing.}$$

And because my solution did not agree with 9819*l*. 11*s*. 3¼*d*. it was condemned to light a pipe. I should be infinitely obliged if, through the medium of your numerous correspondents, you would obtain an answer to the above, not to convince me, for I have made up my mind that mine is right, but to assure so respectable a body of mechanics, to whom I stand opposed, that 9819*l*. 11*s*. 3¼*d*. is wrong.

I am, Gentlemen, Yours very respectfully,

W. P.

Mechanic's Magazine, No. 17, p. 209, December 20, 1823.

(The following issue contains an editorial note saying that seventeen different solutions had been received. The resulting correspondence, too voluminous to be reproduced here, is curious.)

1140. Two things the student needs to know in advance, namely : the intrinsic meaning of a differential equation, and the existence theorems governing the solution in a restricted region. The proof of the latter affords opportunity for expounding, on the hand of this important application, the greatest single method in all science—the Method of Successive Approximations.—W. F. Osgood, *Functions of Real Variables* (1936), p. v.

* Fabian, *Math. Gazette*, vol. 20 (1936), pp. 88-92.

THE THEORY OF COMPLEX NUMBERS.*

By G. Temple.

1. *Problems raised by the introduction of complex numbers.*

The object of this address is to pass in review various theories of complex numbers and to scrutinise them from the standpoint of the teacher engaged in initiating his pupils into this subject. The material of the address is therefore very elementary and well known and roughly a century old, but it is arranged and combined in a way which may perhaps throw some new light on an old problem. It is important to realise at the outset the fundamental issues which are involved; and, although it is undesirable to intrude explicit philosophy into a first lesson on complex numbers, it is well for the teacher to have these metamathematical considerations clear in his own mind.

Broadly speaking, theories of complex numbers fall into two sets—those in which the complex numbers are *described*, and those in which they are *constructed*. In a descriptive theory no attempt is made to define complex numbers or to establish by proof their fundamental properties. Instead, the complex numbers are simply postulated, and their properties are described in a set of axioms. These axioms are the axioms of ordinary real algebra together with the further axioms—"There exists a number i, such that $i^2=-1$. Every element of the algebra has the form $x+iy$, where x and y are real numbers. If $x+iy=0$, then $x=0$ and $y=0$." Now, although this method of direct postulation is adopted without comment in some textbooks, it is highly unsatisfactory if it is advanced as a self-complete theory. Two questions arise at once and demand an answer: What is meant by the "existence" of the number i? and, Is the axiomatic description of complex numbers self-consistent? Until these points are settled the descriptive theory remains incomplete, and these are precisely the two questions raised by every intelligent student.

In a purely descriptive theory there is only one answer to the first question. It is—the number i and complex numbers in general "exist" in the sense that the axioms by which they are described are self-consistent and cannot lead to a contradiction. But this assertion of self-consistency is by no means self-evident and requires proof.

This will be realised more clearly by considering two other algebras, very similar to the algebra C of complex numbers—the algebra N of negative numbers and the algebra D of dual numbers. In N every element has the form $x+\omega y$, where x and y are positive, real numbers; $\omega^2=1$, but $\omega\neq 1$; and if $x+\omega y=0$, then $x=y$. In D every element has the form $x+\omega y$, where x and y are real numbers; $\omega^2=0$, but $\omega\neq 0$; and if $x+\omega y=0$, then $x=0$, $y=0$. In all these

* Presidential Address to the London Branch of the Mathematical Association, 12th December, 1936.

algebras there is a unit ω satisfying an equation of the form $\omega^2=\lambda$, where λ is a real number, viz. -1, 1 or 0. But C and N differ widely in the consequences to be drawn from the equation $x+\omega y=0$, and D differs markedly from C in that in D a product of two elements, say ωa and ωb, can vanish, although neither factor of the product is zero.

It is sometimes stated, explicitly or by implication, that it is self-evident that the axioms of C are self-consistent. It is very difficult to judge this point as a problem of abstract algebra, prescinding from all the geometrical and arithmetical associations of complex number with which we are familiar. But a comparison of C, N and D is very instructive. All three algebras have a similar formal structure, and it therefore seems that the note of self-evident self-consistency should be attached to all if it is attached to one. But the divergence between C and N, and the bizarre character of D make one hesitate to do this, and this hesitation seems fatal to the self-evidence of the self-consistency of C.

If this line of thought is followed so far, it is necessary to advance still further and to attempt to *demonstrate* the self-consistency of C. No *internal* proof seems to have been advanced that the axioms of C cannot lead to a contradiction, and it is therefore necessary to consider *external* proofs in which the self-consistency of C is inferred from the existence of certain logical entities which satisfy all the axioms of C. This line of argument is, in effect, identical with the *constructive* theories of complex numbers in which they are exhibited as the results of certain logical operations with real numbers. There are two constructive theories—in one complex numbers are exhibited as vectors, and in the other as operators. The two theories are frequently combined or confused in the methods of exposition adopted in certain standard textbooks,* but they are essentially distinct and will here be considered separately.

2. *Complex numbers as vectors.*

A vector may be defined as a directed straight line, specified by its terminal points in a definite order, *e.g.* AB. The length of a vector AB is the distance of B from A. Two vectors AB, CD are equal if AB is equal and parallel to CD and in the same sense. The sum of two co-initial vectors AP, AQ is the directed diagonal AR of the parallelogram $APRQ$. The sum of any two vectors AB and CD is then defined to be the sum of AB and a vector AE equal to CD. The product of a vector AB and a real number r is a vector AC such that the length of AC is $|r|$-times the length of AB, and AC has the same or the opposite sense as AB according as r is positive or negative. A null vector has zero length.

From these definitions it follows that if ι_x and ι_y are vectors OA, OB of unit length along the coordinate axes OX, OY, and if P has

* See, for example, *The Teaching of Algebra*, by Sir Percy Nunn, section vii, (1926). Compare the theory of addition in § 2 and the theory of multiplication in § 3.

the coordinates (x, y), then the vector OP is equal to $x\iota_x + y\iota_y$, and it may be represented by the ordered number pair (x, y), consisting of the x- and y- components of OP. The addition of two vectors then provides an instance of the axioms relating to the addition of two complex numbers, but it is far otherwise with the problem of multiplication. The kind of multiplication which must be adopted in order to satisfy the axiom of C is such that

$$\iota_x^2 = \iota_x, \quad \iota_x\iota_y = \iota_y, \quad \iota_y^2 = -\iota_x.$$

This kind of multiplication is not "scalar" multiplication or "vector" multiplication as commonly defined; and it is obviously essentially non-vectorial in character, as the two vectors ι_x and ι_y behave quite differently from one another. Nor is the situation improved by writing $\iota_x = 1$, $\iota_y = i$.

It follows that the representation of complex numbers as vectors is inadequate. Of course this fact is well known, and reference to standard expositions discloses two methods of procedure. In one the law of multiplication is imposed by an *ad hoc* definition. In the other it is obtained by an appeal to the second type of constructive theory in which complex numbers are represented as operators. The *ad hoc* definition of multiplication inevitably raises queries from the student, queries which are not stifled by the reply that it is the only definition which makes multiplication commutative, associative, and such that the vanishing of a product implies the vanishing of at least one factor. Nor is it satisfactory to represent complex numbers as vectors in order to define addition, and to represent them as operators in order to define multiplication.* It therefore remains to consider a thoroughgoing *operational* theory, such as that propounded by Hamilton.†

3. *Complex numbers as operators.*

In Hamilton's theory a complex number p is in effect defined as the operator which converts one vector α into another vector β, and this relation is symbolised by writing

$$p\alpha = \beta, \quad \text{and} \quad p = \beta/\alpha.$$

The modulus r and the argument θ of p are respectively defined as the ratio of the lengths of β and α, and as the angle between their directions. These two numbers completely specify a complex number p which can therefore be conveniently expressed for some purposes in the form (r, θ), which has a certain popularity with electrical engineers.

Two complex numbers $p = \beta/\alpha$ and $q = \delta/\gamma$ are said to be equal if the triangles OAB and OCD are similar and similarly situated, where $OA = \alpha$, $OB = \beta$, $OC = \gamma$, $OD = \delta$. With this introduction of operational concepts the definition of both the sum and product of

* Cf. Nunn, *loc. cit.*

† *Elements of Quaternions*, Sir W. R. Hamilton (Longmans, Green & Co., 1866, out of print).

two complex numbers imposes itself quite naturally, first in two special cases and secondly in the general case.

The sum of ρ/α and σ/α is naturally defined to be $(\rho+\sigma)/\alpha$, and the product of β/α by γ/β is naturally defined to be $(\gamma/\beta).(\beta/\alpha)=\gamma/\alpha$. (*N.B.*—Nothing is said about the product $(\beta/\alpha).(\gamma/\beta)$, but it will appear later that this is also equal to γ/α). In the general cases we define the sum and product as follows :

If $$\epsilon=(\delta/\gamma)\alpha,$$
then $$\beta/\alpha+\delta/\gamma=\beta/\alpha+\epsilon/\alpha=(\beta+\epsilon)/\alpha.$$
If $$\epsilon=(\delta/\gamma)\beta,$$
then $$(\delta/\gamma)(\beta/\alpha)=(\epsilon/\beta)(\beta/\alpha)=\epsilon/\alpha.$$

To obtain the standard representation of complex numbers it is necessary to introduce two special complex numbers—the identical operator $I=\alpha/\alpha$, which leaves unaltered any vector to which it is applied, so that $I\beta=\beta$, for any vector β ; and the operator $J=\iota_y/\iota_x$, where ι_x and ι_y are the unit vectors along the axes, so that $J\alpha=\beta$, where β is obtained from α by a counter-clockwise rotation through a right angle. It follows from the definition of the equality of operators that

that $$J=(-\iota_x)/\iota_y=\iota_x/(-\iota_y),$$
whence $$J^2=(\iota_y/\iota_x)(\iota_x/-\iota_y)=(\iota_y/-\iota_y)=-I.$$

It is usual to replace I by 1, the unit of the real number system, and to replace J by i in mathematics, or by j in electrical engineering. Actually the long debate between these rival schools can only be concluded by replacing J by k (as in quaternionic theory)—the only symbolism which indicates that the axis of rotation of J is the z-axis !

However, adopting the usual mathematical notation, it appears that the position of a point P can be specified either by the vector OP or by the complex number

$$OP/OA=(x\iota_x+y\iota_y)/\iota_x=xI+yJ=x+yi.$$

It then follows from the preceding definitions that addition is commutative and associative.

To obtain the representation of a complex number in terms of its modulus r and argument θ, let $\theta=\frac{1}{2}\pi m$, where m is a real number. Then by an extension of the usual theory of indices, the symbol i^m can be defined to be the operator which rotates a vector through m right angles without changing its length ; the real number r, regarded as an operator, stretches a vector in its own direction in the ratio $r:1$ (if r is positive). Hence there arises the relation

$$x+yi=ri^m,$$
where $$x=r\cos\tfrac{1}{2}\pi m,$$
and $$y=r\sin\tfrac{1}{2}\pi m.$$

Hence $$i^m = \cos \tfrac{1}{2}\pi m + i \sin \tfrac{1}{2}\pi m = \operatorname{cis} \tfrac{1}{2}\pi m,$$
in the abbreviated notation of Harkness and Morley.

It is now easily proved that multiplication is commutative and associative, and that addition is distributive. The proof of the commutativity of multiplication shows, as stated above, that
$$(\beta/\alpha) . (\gamma/\beta) = \gamma/\alpha.$$

The operational representation of complex numbers which has just been sketched presents many advantages over the vectorial representation. The definitions of addition and multiplication both arise as almost inevitable developments of the theory: the operational concept seems to be fully in accord with the views of engineers and physicists on the nature of complex numbers: and the theory is easily extended by Hamilton's methods from two dimensions to three. For the professional mathematician there is the additional advantage of an early acquaintance with the concepts of operators and groups, which dominate so much of contemporary pure and applied mathematics.

4. *Illustrations of the operational character of complex numbers.*

(*a*) An elementary example is provided by the theory of alternating currents. The impedance Z of a circuit is actually defined as the ratio of the electromotive force vector E to the current vector C; and for a coil of resistance R, inductance L and capacity K, the expression for Z is
$$Z = E/C = i\omega L + R + 1/i\omega K,$$
where ω is the pulsatance of the current.

(*b*) An interesting application is to determine the polar components of a given vector. If α is any complex number and ν any complex number of unit modulus, and if
$$\alpha/\nu = p + is,$$
then p and s are the components of α respectively parallel and perpendicular (senkrecht) to ν. The complex number representing the velocity of the point P with coordinates (x, y) at time t is
$$\begin{aligned} \dot{z} &= \dot{x} + i\dot{y} = d(r \operatorname{cis} \theta)/dt \\ &= \dot{r} \operatorname{cis} \theta + i\dot{\theta} r \operatorname{cis} \theta. \end{aligned}$$
Now the unit complex number ν along the radius vector is $\operatorname{cis} \theta$.

Therefore $$\dot{z}/\nu = i + i\dot{\theta} r,$$
whence, in polar coordinates, the components of the velocity are $\dot{r}$ and $r\dot{\theta}$.

Similarly $$\ddot{z} = \ddot{r} \operatorname{cis} \theta + 2i\dot{\theta}\dot{r} \operatorname{cis} \theta - \dot{\theta}^2 r \operatorname{cis} \theta + i\ddot{\theta} r \operatorname{cis} \theta$$
and $$\ddot{z}/\nu = (\ddot{r} - r\dot{\theta}^2) + (2\dot{r}\dot{\theta} + r\ddot{\theta})\, i,$$
whence the polar components of the acceleration are
$$\ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad d(r^2\dot{\theta})/rdt.$$

(c) $w=u+iv$ is an analytic function of $z=x+iy$ if dw/dz exists as a unique limit $\lambda+i\mu$ independent of the argument of δz. Now

$$\frac{dw}{dz}=\left(\frac{du}{ds}+i\frac{dv}{ds}\right)\Big/\operatorname{cis}\psi,$$

where δs and ψ are respectively the modulus and argument of δz. Hence the vector with components du/ds, dv/ds along the axes of x and y has components λ, μ in directions making an angle ψ with the axes of x and y respectively. Applying this argument to two increments $\delta_1 z$, $\delta_2 z$ in perpendicular directions, it follows that the vectors with components $(du/ds_1, dv/ds_1)$ and $(du/ds_2, dv/ds_2)$ along the axes are equal in magnitude and perpendicular in direction. Hence

$$\frac{du}{ds_1}=\frac{dv}{ds_2} \quad \text{and} \quad \frac{dv}{ds_1}=-\frac{du}{ds_2}.$$

In these equations δs_2 is derived from δs_1 by a counter-clockwise rotation through one right angle. These rotations are the familiar Riemann-Cauchy equations expressed in invariant form, and it is this form which is required in applications to two dimensional problems in electrostatics. (For, if u is the potential, δs_1 is along the outward normal to a conductor, and δs_2 along the boundary of the conductor, then the line-density of the charge is

$$du/4\pi\, ds_1=dv/4\pi\, ds_2,$$

and the total charge is $(1/4\pi)\int(dv/ds_2)\,ds_2$ or $(1/4\pi)$ times the change in the value of v on going round the conductor. There seems to be no generally accepted name for the function v in electrostatics. Clearly the equation $v=$constant gives the lines of force, and the difference in the values of v at two points A and B gives the total normal induction or electrostatic flux across any curve joining A and B in empty space. Perhaps v might therefore be called the *induction* function.) G. T.

1141. Let it be said that no one can *teach* the student the Theory of Functions. For the Theory of Functions is a habit of thought, not a set of rules to be applied like the formulas of differentiation.—W. F. Osgood, *Functions of Real Variables* (1936), p. iv.

1142. " . . . you can *reason* for all you are worth—and be entirely wrong. But what you really *feel* is never wrong, if you know what I mean."

" Yes, I know what you mean. Instinct, intuition and that sort of thing are better than reason, logic and mathematics."

" A lot better. You can prove anything, with figures, and reason can lead you anywhere; but if you've got a real strong *feeling* about something, deep-seated and unshakable, it is bound to be right."—P. C. Wren, *Bubble Reputation* (John Murray), p. 228. [Per Mr. J. P. McCarthy.]

1143. [General Sir John Burgoyne] To high intellectual power he added the firmness of a reasoner who holds that there can be no sect in mathematics, and that opinions carefully formed must not be dominated by mere results. —A. W. Kinglake, *The Invasion of the Crimea*, Cabinet edition, iii, p. 392.

MATHEMATICAL NOTES.

1242. *Note on integration.*

1. An interesting point arose from an error made in working the following example.

Find the volume obtained by rotating the area common to

$$y=x^2, \quad y=x$$

about the latter.

The line and parabola intersect at (1, 1). Take a point $P(x, y)$ on the parabola, and let P' be a neighbouring point on the curve (Fig. 1). Then the perpendicular from P to the line,

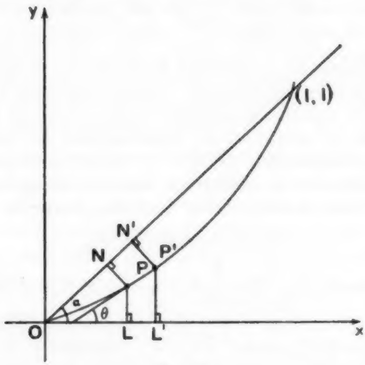


Fig. 1.

$$PN=|y-x|/\surd 2$$

and

$$NN'=LL' \sec\theta \cos(\theta-\alpha)$$

$$=LL'(1+2x)/\surd 2.$$

$$\text{Volume}=\lim \Sigma\pi \,.\, PN^2 \,.\, NN'$$

$$=\frac{\pi}{\surd 2}\int_0^1 (y-x)^2(1+2x)\,dx$$

$$=\pi/15\surd 2.$$

The error made was in taking

$$NN'=LL' \sec\alpha.$$

$$\text{Volume}=\pi\int_0^1 \surd 2 \,.\, (y-x)^2\,dx$$

$$=\pi/15\surd 2 \text{ as before.}$$

2. The matter was investigated further, and two interesting facts were discovered.

(*a*) The incorrect method *always* gives the correct answer, whatever the curve and the line.

Let the curve be $y=f(x)$ and the line $y=mx+c$, and let them intersect at $x=a$, $x=b$.

Then $$PN=(y-mx-c)/\surd(1+m^2).$$

The correct value of NN' is

$$LL'\{1+mf'(x)\}/\surd(1+m^2)$$

and the incorrect value of NN' is

$$LL'\surd(1+m^2).$$

If I_1 is the correct and I_2 the incorrect integral

$$\begin{aligned} I_1-I_2 &= \int_a^b \frac{(y-mx-c)^2}{(1+m^2)}\left[\surd(1+m^2)-\frac{\{1+mf'(x)\}}{\surd(1+m^2)}\right]dx \\ &= m(1+m^2)^{-\frac{3}{2}}\int_a^b \{f(x)-mx-c\}^2\{f'(x)-m\}\,dx \\ &= m(1+m^2)^{-\frac{3}{2}}\left[\tfrac{1}{3}\{f(x)-mx-c\}^3\right]_a^b \\ &= 0, \end{aligned}$$

since $f(x)-mx-c$ vanishes at $x=a$ and at $x=b$.

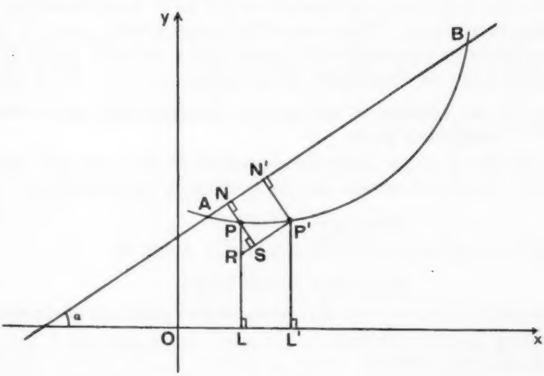


FIG. 2.

(*b*) The method involving the error gives the same result as the correct method whatever be the function of the perpendicular PN that is taken.

In Fig. 2, let PN, the perpendicular from P to the line, be p,

$$P'N'=p+\delta p.$$

If $P'R$ is parallel to AB, then $PS=\delta p$.

$$\angle RPS=\alpha, \quad \angle PSR=90^\circ.$$

Thus $$RS=\delta p \tan . \alpha.$$

The incorrect value taken for NN' is $LL' \sec\alpha=P'R$.

Thus the difference between the correct and incorrect values of NN' is

$$\begin{aligned} P'R-NN' &= P'R-P'S \\ &= RS \\ &= \delta p \,.\, \tan\alpha. \end{aligned}$$

Hence if $\phi(p)$ is any integrable single-valued function of p, the difference between the two integrals is

$$\begin{aligned} &\int_A^B \phi(p) \,.\, \tan\alpha \,.\, dp \\ &= \tan\alpha \Big[\chi(p)\Big]_A^B \\ &= 0, \end{aligned}$$

since at A and B, $\chi(p)=\chi(0)$.

Hence the two integrals are always equal.

3. Thus taking the " incorrect " value for the increment produces the correct result in every case. It simplifies the work, as is seen in the example of section 1. If in this example the area of the surface of revolution had been required, it could have been found by using this other increment. There are other applications, such as finding the area between a complicated curve and a straight line where the result is of value in simplifying the work. W. E. Egner.

1243. *On the relation of an analytic function of z to its real and imaginary parts.*

Let $f(z)$ be a given analytic function of the complex variable $z=x+iy$. Such a function can be written in the form

(1) $$f(z)=f_1(z)+if_2(z),$$

where $f_1(z)$, $f_2(z)$ are real when z is real. Again if

(2) $$f(z)=\phi(x, y)+i\psi(x, y),$$

where ϕ and ψ are real, we can propose the problem of determining $f(z)$ where $\phi(x, y)$ is given.

Let $\bar{z}=x-iy$. Then

(3) $$f(z)=\phi\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)+i\psi\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

This is an identity. Putting $\bar{z}=z$, we then get

(4) $$f_1(z)+if_2(z)=f(z)=\phi(z, 0)+i\psi(z, 0).$$

Therefore

(5) $$f_1(z)=\phi(z, 0), \quad f_2(z)=\psi(z, 0).$$

In the particular case where $f(z)$ is real when z is real, *i.e.* if $f_2(z)\equiv 0$, we see that

$$f(z)=\phi(z, 0).$$

Example. If $\phi=\sin x\cosh y$, $f(z)=\sin z$.

It is evident from (5) that no such simple result is available when $f_2(z)$ does not vanish identically for then it is necessary to know $\psi(x, y)$. To deal with this case we observe that ϕ and ψ are conjugate functions and therefore from (1) and (2):

$$\begin{aligned} f_1'(z)+if_2'(z)=f'(z)&=\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}\\ &=\phi_1(x, y)-i\phi_2(x, y) \quad \text{say}\\ &=\phi_1(z, 0)-i\phi_2(z, 0) \quad \text{from (4).}\end{aligned}$$

Thus
$$f(z)=\int[\phi_1(z, 0)-i\phi_2(z, 0)]\,dz.$$

Example. $\phi(x, y)=\sin x\cosh y+2\cos x\sinh y+x^2-y^2+4xy$,

$$\begin{aligned} f(z)&=\int(\cos z+2z-2i\cos z-4iz)\,dz\\ &=\sin z-2i\sin z+z^2-2iz^2.\end{aligned}$$

The case of determining $f(z)$ when $\psi(x, y)$ is given is easily treated on the same lines.

If both $\phi(x, y)$ and $\psi(x, y)$ are given, $f(z)$ can be written down by using (4).

As an interesting consequence of (1), the *real or conjugate* roots of the equation $f(z)=0$ must be zeros of the highest common factor of $f_1(z)$, $f_2(z)$, and can therefore be removed from the equation by algebraic operations when $f(z)$ is a polynomial in z.

L. M. MILNE-THOMSON.

1244. *A note on Bessel's function of order zero.*

The following comparison of

$$\frac{d^2y}{dt^2}+a\frac{dy}{dt}+by=0, \quad \text{..............................(1)}$$

and
$$\frac{d^2y}{dt^2}+\frac{1}{t}\frac{dy}{dt}+by=0, \quad \text{..............................(2)}$$

where a, b are constants, may be useful for teaching purposes.

Taking $b>\frac{1}{4}a^2$, $y=y_0$, $y'=0$ at $t=0$, the respective solutions of (1) and (2) are

$$y=y_0e^{-\frac{1}{2}at}\cos(\beta t-\epsilon), \quad \text{..................................(3)}$$

$$y=y_0J_0(\sqrt{b}t), \quad \text{..(4)}$$

with
$$\beta=\sqrt{b-\tfrac{1}{4}a^2} \text{ and } \epsilon=\arctan(a/2\beta).$$

The motion of a dynamical or other system represented by (2) is oscillatory, the time displacement curve being given at (4). When

is very small, $1/t$ is very large as also is the damping in (2). As t increases the damping decreases, which corresponds to a reduction of a in (1). Reduction in damping with increase in t introduces a decrease in the distance between successive zeros on the time axis, which is equivalent to a rise in frequency.

If (1) is the differential equation of a mechanical system comprising mass m, resistance to motion r per unit velocity, and a spring of stiffness s dynes per cm., then $a=r/m$ and $b=s/m$. Assuming that r were due to fluid damping, which might be made to vary approximately as $1/t$, a time displacement curve of the form $J_0(b^{\frac{1}{2}}t)$ would be obtained (provided $1/t=r/m$). For a closed electrical circuit comprising inductance L, resistance R, and capacitance C in series, $a=R/L$, $b=1/LC$. In this case to obtain the Bessel function curve we must have $R/L=1/t$, or $R=L/t$. N. W. McLachlan.

1245. *A note on* $x^2+y^2=N$.

Given that $h^2+k^2=N$ where h, k and N are rational, to find a formula for other rational solutions of $x^2+y^2=N$.

Interpret $x^2+y^2=N$ as a circle. Then (h, k) is a point of this circle. Any line through (h, k) is of the form $y=mx+(k-mh)$. For the abscissae of the points where this line meets the circle we have

$$x^2+\{mx+(k-mh)\}^2=N,$$

or

$$x^2(1+m^2)+2m(k-mh)x+\{(k-mh)^2-N\}=0.$$

But one root of this is $x=h$, therefore the other must be

$$x=\frac{(k-mh)^2-N}{h(1+m^2)}=\frac{(m^2h-2mk-h)}{(1+m^2)}. \quad \text{.................(1)}$$

Interchanging h and k, and writing $1/m$ in place of m, the corresponding value for y is

$$y=\frac{(k-2mh-m^2k)}{(1+m^2)}. \quad \text{............................(2)}$$

It can easily be verified that

$$(m^2h-2mk-h)^2+(k-2mh-m^2k)^2=(h^2+k^2)(1+m^2)^2.$$

If m is rational then the expressions (1) and (2) must be rational, conversely the line joining two points whose coordinates are rational must have a rational m. Hence (1) and (2) give, for rational m, all rational solutions of $x^2+y^2=N$. Writing $m=p/q$, we obtain the solutions in a more convenient form :

$$x=(p^2h-2pqk-q^2h)/(p^2+q^2),$$
$$y=(q^2k-2pqh-p^2k)/(p^2+q^2),$$

where p and q are integers prime to each other.

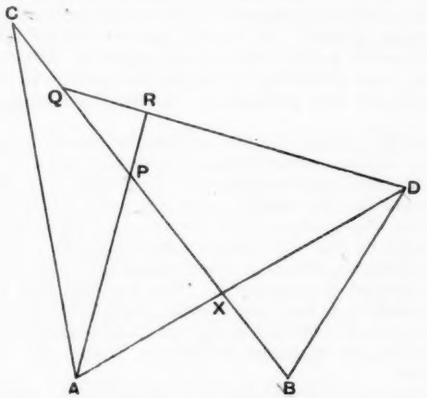
The method applies of course to any quadratic form in x and y.

J. Clemow.

1246. *A proof of an elementary theorem in geometry.*

I read with interest Mr. Piggott's review of two recently published school geometries, particularly his strictures on the proofs of the theorem : Given angle $ACB=ADB$, C and D being on the same side of AB, prove that $ACDB$ are concyclic.

The following proof of the above theorem, I think, avoids Mr. Piggott's objections, and may interest teachers of elementary geometry. I cannot claim originality for it : I worked it on the model of that excellent proof, given by Herr Lietzmann in *Altes und Neues vom Kreis* (Teubner), of the converse proposition of the cyclic quadrilateral property.



Given : $\widehat{ACB}=\widehat{ADB}=a$.

To prove : A, B, C, D concyclic.

Proof : Clearly the triangles CXA, DXB are similar.

Let $\widehat{CAD}=b=\widehat{DBC}$. Let a be less than b.

Take a point on CB such that $\widehat{PAC}=\widehat{PCA}$.

Take a point Q on BC such that $\widehat{QDB}=\widehat{QBD}$, and let AP, QD intersect at R. Then $\widehat{RAD}=b-a=\widehat{RDA}$. Thus RAD is isosceles.

Now the bisectors of the angles BQD, ARD and RPB meet at a point, X say.

But as BQD is isosceles, the bisector of $\widehat{BQD}$ is the perpendicular bisector of BD, and is thus the locus of points equidistant from B and D.

Thus X is equidistant from B and D, and similarly X is equidistant from A and C, and from A and D.

Therefore X is equidistant from all four points A, B, C, D, which are therefore concyclic. JOHN E. BLAMEY.

REVIEWS.

Relativity Theory of Protons and Electrons. By Sir A. S. EDDINGTON. Pp. vi, 336. 21s. 1936. (Cambridge)

Anyone attempting to write a notice of this book might well try to imagine the thoughts of a reviewer for about Volume *minus* 200 of the *Gazette* setting out to review the new work, *Philosophiae Naturalis Principia Mathematica* by I. Newton. I do not mean to beg any questions about the relative importance of the books. But both reviewers would be overwhelmed by the wealth of new ideas presented to them and by the vastness of the deductions from so few premises. Both, too, would be compelled by this circumstance to approach their tasks with the utmost humility. The reviewer in Volume 21 can bring himself to the performance only by proposing to do no more than sketch briefly something of what Eddington has done ; he shrinks from any attempt to assess its value. However, he consoles himself with the thought that the value of such a work, which is throughout concerned with the most fundamental aspects of our knowledge of the physical world, will probably in any case be assessed only very gradually by the combined judgment of natural philosophers.

There is actually a profound difference between Newton's and Eddington's starting points. Newton started from a few simple empirical laws. Eddington, on the other hand, tries to construct a theory with no empirical elements at all. He summarises his outlook in the last section of the book by saying, " An intelligence, unacquainted with our universe, but acquainted with the system of thought by which the human mind interprets to itself the content of its sensory experience, should be able to attain all the knowledge of physics that we have attained by experiment." One can only think that, supposing the fate of the book to be the worst possible, if physicists were to reject every idea contained in it, it could not at any rate lose its fascination as a monumental attempt by one particular intelligence to show how this attainment might be reached.

I think that every successful theory in mathematical physics appears in retrospect to have an element of luck in it. This certainly seems to be the case with current quantum theory. For, on the one hand, it insists on calculating " observables " and also on the fact that the process of observation disturbs the system observed. On the other hand, most of its calculations on atomic systems, in spite of their successful issue, make no mention of any observing mechanism, *i.e.* they ignore the " rest of the universe ". Some aspects of this dilemma have been discussed by Bohr. But now the current theory of the " rest of the universe ", *i.e.* of macroscopic phenomena, is relativity theory. If therefore we want to repair such apparent weaknesses in the basis of quantum mechanics it appears that we must somehow link it up with relativity theory. To do this is Eddington's aim, only it has resulted, not in a patching-up of existing theories, but in their complete re-casting.

His book is in two parts ; Part I consists of ten chapters on " Wave-Tensor Calculus ", and Part II consists of six chapters on " Physical Applications ". Since the mathematical theory developed in Part I is novel as regards any application to physics, one cannot describe the derivation of the physical results without first summarising this theory.

It is significant that Eddington finds his new advances to depend on modern algebra, for this appears to be in keeping with the general trend of mathematical physics. Relativity theory did away with mechanical models, but to some extent substituted geometrical models in their place. Now the newer theories of Eddington and of Milne, widely as they differ in aims and methods, both tend to dispense with geometrical models. In Eddington's case the sug-

gestion for departing from the geometrical linkage comes from the fact that Dirac's wave equation exhibits a type of invariance foreign to relativity theory. As is well known, this fact led to the development of the *spinor* calculus by van der Waerden, Veblen and others, and it has been shown only this year (1937) in papers by E. T. Whittaker and by H. S. Ruse how it can in fact be related to the geometrical scheme of general relativity or to natural extensions of it. But Eddington sees in the fact the need for a more fundamental extension of current ideas. He denies the elementary geometrical displacement, described by a vector (dx^p), its fundamental place in the theory, and, instead of the corresponding space-tensors, develops a theory of wave-tensors, depending on a fundamental wave-vector χ_μ. Tensor transformation laws are used, and the distinction between covariant, contravariant and mixed tensors is retained, but instead of the ordinary transformation matrices given by the coordinate transformation, more general transformation symbols are employed. These new tensors are the " wave-tensors ".

Eddington starts by setting up a 16-fold frame in the following way: he considers four symbols E_1, E_2, E_3, E_4 which satisfy

$$E_\mu E_\nu = -E_\nu E_\mu (\mu \neq \nu), \quad = -1 (\mu = \nu), \quad (\mu, \nu = 1, 2, 3, 4).$$

Any continued product of any number of these symbols reduces to one of 16 different forms E_λ $(\lambda = 1, 2, \dots, 16)$. Any linear function $\Sigma_1^{16} t_\lambda E_\lambda$, whose coefficients t_λ are real or complex numbers, is called an *E-number*. So the sum, difference, or product of any number of E-numbers is itself an E-number. The sixteen E_λ's form a frame of reference in much the same way as do the symbols **i**, **j**, **k**, of quaternion theory. They are in fact, as Eddington points out, the special case corresponding to four basic symbols of a general class of symbols discussed by Clifford in 1878. The reason for selecting 4 as the basic number is not empirical, but is later shown to depend on the ultimate meaning of measurement. For the most elementary measurement consists of the comparison of two relations, each relation involving two relata, thus involving in all precisely four relata. The four-dimensional character of space-time is made to depend ultimately on the same circumstance.

An essential part of every system of mathematical physics is its *transformation theory*. In the present system the law of transformation of a " mixed wave-tensor " is given by the theorem that, if q, q^* are any symbols such that $qq^* = q^*q = 1$, then the symbols $E_\lambda' = qE_\lambda q^*$ $(\lambda = 1, 2, \dots, 16)$ constitute a new set of 16 E-symbols having the same properties as the initial set. This transformation is therefore regarded as a *rotation* of the 16-fold frame. Further, if any physical system is represented as an E-number, and the E's are submitted to this transformation, then the system is regarded as being rotated without distortion, and is said to undergo a relativity transformation. These ideas are exactly analogous to the transformation from one Galilean frame of space-time to another in special relativity theory.

Eddington then shows how the E-symbols can be represented by " four-point " matrices. This representation is not considered a fundamental part of the theory, but rather as an aid to the manipulation. Next, in view of prospective connections with quantum theory, it is not surprising to find introduced a general definition of eigensymbols and eigenvalues. Given any symbol X, if a symbol ϕ can be found such that $X\phi = \alpha\phi$, where α is a number, real or complex, then ϕ is an eigensymbol, and α an eigenvalue of X. The properties and calculation of these are discussed.

Rotations of the sort described are then further analysed. It is shown how, by a proper choice of labelling the E_λ'*s*, the quantities t_1, t_2, t_3, t_4, t_5 can be regarded as coordinates in a 5-dimensional euclidean space, *i.e.* as a " space-vector ", this space being a sub-space of the 16-dimensional space of the

16 components t_λ. The same analysis shows that the remaining components of the mixed wave-tensor can be regarded as constituting an antisymmetrical space-tensor together with an invariant. The 5-dimensional sub-space requires physical interpretation. Any set of equivalent points in this space, *i.e.* points which are transformable into each other by rotations, lie on a hypersphere, centre the origin. The surface of such a hypersphere is taken as the space-time continuum, and the change from one such surface to another is taken to correspond to a transformation of " gauge ". It must be remembered that we are dealing, as in all applications of relativity theory, with a smoothed-out universe, only here the smoothing is the most drastic which it is possible to make and still preserve the essential feature of a *curvature* of space-time. The effect of this is that the theory is applicable to an Einstein or a de Sitter universe, which is sufficient to give finally the desired relation to the large-scale properties of the universe.

The simple wave-equation is then introduced. This is of the form $H\psi=0$, where H is a mixed wave-tensor, and may be taken to define ψ, a wave-vector. Dirac's wave-equation is found to be a special case, and its peculiar type of invariance is an immediate consequence of the definition in Eddington's theory.

A chapter on *reality conditions* brings in some very fundamental ideas. They are introduced by considering how many of the representative matrices may be taken with all real elements. The conditions take the place of the Hermitian condition in current wave-mechanics. The first result is to deduce the usually assumed signature of space-time. The next may be crudely summarised as follows: the theory supplies right-handed and left-handed E-frames, and corresponding vectors cannot be transformed into each other by relativity rotations. So these vectors are identified with what are called the " complete stream vectors " of positive and negative charged particles. Then it is shown that, with this identification, if a region of space-time contains only positive charged particles and if the radius of curvature is assigned a direction n, say, then if it contains only negative charged particles the radius has the direction $-n$, and if it contains only neutral particles its radius has direction $i\,n$. It is further shown how certain components of the complete stream vector represent a *spin*, and how it is possible to represent particles with positive or negative charges combined with positive or negative spins, and also neutral particles.

The next step is in the direction of dealing with composite systems, and is guided by the analogy of the analysis into motion of the centre of mass (external motion) and the motions of the parts relative to the centre of mass (internal motions) used in ordinary dynamics. The important point is made that there is no question of the internal motions being invariant under a Lorentz transformation, for these motions are relative to an assigned, not an arbitrary, standard of rest. It is looked after in the mathematics by using a *strain vector* to represent internal properties. This is got from a *covariant* wave-tensor in the same way as a complete space-vector is got from a mixed wave-tensor. If it is plotted in the 16-dimensional space, then when a tensor transformation is applied it is displaced relatively to the space-vectors. This relative displacement is shown to have 10, not 16, degrees of freedom, and the corresponding 10-dimensional space is called the *phase-space* of the strain vector. The name is appropriate because the displacements being relative to space-vectors are not relativity transformations, but represent intrinsic deformations of the system, *i.e.* different points of the phase-space may be said to represent different states of the system. In applications the phase-space is supposed to be the seat of a probability distribution.

The next chapters give the definition of a divergence operator, and its use to express the conservation of total probability. This leads on to the formulation of the differential wave-equation. It is solved in detail, by the methods of the present calculus, for the hydrogen atom, the same results as those of Dirac's theory being derived.

The final stage of the mathematical theory is the development of the theory of *double wave-vectors*. It bears some relation to the feature of Schrödinger's theory which required the addition of three dimensions for every particle introduced. In Eddington's theory there are more dimensions than this, but he gives reasons, connected with the fundamental meaning of measurement already alluded to, for not having to consider in detail the interaction of more than two particles at a time. His double wave-vectors are introduced from a consideration of the combinations of probabilities. Such a vector is obtained by simply multiplying two simple wave-vectors belonging to two independent systems. The systems being independent, they have each their 16 E-symbols, so the double frame has 256 dimensions. It is shown that the corresponding phase-space has 136 dimensions, a number which turns out to have great significance. Since it is at this stage that double wave-vectors make their appearance, the connection with the fundamental tensors of the second rank $g_{\mu\nu}$, $T_{\mu\nu}$ of relativity theory is at once suggested.

This brings us to Part II, in the course of which Eddington deduces from the theory values for the mass-ratio of the proton and electron, the ratio of their magnetic moments, the pressure in a degenerate gas, the fine-structure constant, the total number of particles in the universe (or at any rate the relation between this number and the gravitational constant), the speed of recession of the nebulae and the cosmical constant, and also the inverse square law of electric force and a theory of the Exclusion Principle. Eddington was the first, I think, to stress the fact that quantities like the mass-ratio of the proton and electron and the fine-structure constant are dimensionless functions of the fundamental constants of physics, *i.e.* they are pure numbers. The next step was to argue that any pure number at the basis of physics should have a purely mathematical significance, and that an adequate mathematical theory of the fundamental parts of physics should produce these numbers. I think that most natural philosophers (I cannot find a better word in this connection) would grant the validity of this argument, whether or not they would follow Eddington any further in his particular attempt to supply the mathematical theory. However, the fact that his theory does appear to give very closely the observed values of *all* such numbers now known to physics provides a good *prima facie* case for its general correctness.

In a short space one cannot summarise the derivations of these results. Anyhow, full qualitative descriptions of them have been given by Eddington himself in his more popular lectures and writings* (*The Expanding Universe*, 1933; *New Pathways in Science*, 1935). As an example, however, one might take the problem of the mass of an elementary particle. The steps in the work appear to be roughly these: we cannot assign any observable properties to a single isolated particle, but must allow it to interact with some other system before we can observe it. The observable properties therefore belong to the combined system of particle *plus* reference object. This should be represented mathematically by using double wave-tensors for the combined system. But as a matter of fact current theory does not explicitly mention the reference object. Therefore the probabilities by which the mathematics is interpreted must formally have the appearance of being independent for

* Since this review was written there has appeared in a Supplement to *Nature* (1937, June 12) a discussion by various authors of the philosophical bearing of this type of work. (*Added in proof.*)

the particle and reference system. It is shown that this form can be obtained only if the elements of phase-space of the particle, the reference object, and the combined system, satisfy a certain relation. This relation involves the " energy operators " and so the masses. It turns out that it can be satisfied only if the mass of the elementary particle has one of two particular values, which are given as the roots of a certain quadratic equation. The ratio of these roots is 1847·6, so it is natural to identify the two permissible masses with those of the proton and electron. The circumstance that these two masses have this particular ratio therefore emerges merely as the condition that current theory can be legitimately used!

If this book shows the possibility of a purely deductive theory, based only on the mode of working of the mind, it can scarcely fail to mark an epoch. But that one cannot predict. It is safe to predict that it will long be studied with profound interest by mathematical physicists. In 1923 Eddington brought out his *Mathematical Theory of Relativity* and in 1926 his *Internal Constitution of the Stars*, and the author of almost every paper subsequently published on the respective subjects of these books has treated them as the standard works of reference and very frequently indeed as the starting point of his own researches. The *Relativity Theory of Protons and Electrons* is assured of a similarly illustrious career. W. H. McCrea.

Geschichte der Elementar-Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter. III. Proportionen, Gleichungen. By J. Tropfke. Third edition. Pp. 239. RM. 10. 1937. (Walter de Gruyter, Berlin and Leipzig)

The first thing that calls for notice in relation to this volume is the same as would strike the reader of the preceding volume (II), which bears the date 1933, namely the great increase in its size in comparison with the same volume of the second edition. In this case the number of pages has increased from 151 to 239, or by more than 50 per cent., and the number of footnotes from 538 to 807. While the history, already a classic, thus becomes more and more comprehensive, there is a danger in the issue of the third edition being spread over so long a period. The first volume having appeared in 1930, the second in 1933 and the third in 1937, it is likely, indeed inevitable, that, if the remaining four volumes come out at such long intervals, the references to the literature of the subject in the early volumes will need supplementing, even if new discoveries or researches do not greatly affect the substance of the history, before the final volume with the indices sees the light. In the meantime the elaborate indices in Volume VII of the second edition are made to serve for the third edition by means of ingenious tables in which the corresponding pages and note-numbers in the two editions are shown side by side.

The whole of the present volume is taken up by the history of " Proportions " and " Equations ", the former occupying only 22 pages. The reason is that proportion has gradually come to occupy in original works and in textbooks less and less space. This is due to the gradual development of algebraical notation and the process by which algebra has become independent of geometry after being, in the time of the Greek pioneers, limited (owing to the want of notation) to what could be effected by geometry, or what has been called the " geometrical algebra ". This meant that, subject to one qualification, the Greeks could not deal with more than three dimensions, or equations of degree higher than the third. They could solve equations of the first and second degrees by means of the straight line and circle, and they solved some cubic and some biquadratic equations by means of conics or higher curves, or their equivalent. Their only means of transcending these limits lay pre-

cisely in the use of proportions and in the possibility of compounding (and otherwise transforming) ratios to any extent. This is why the theory of proportions plays an infinitely greater part in their mathematics.

The history of equations is treated under the following heads: general historical survey, the distinction between the known and the unknown quantities, technical terms, equations of the first degree (i) with one unknown, (ii) with more unknowns, equations of the second degree, "reciprocal" equations, quadratic equations with more than one unknown, equations of the third degree and solutions of the same by algebra, geometry and trigonometry respectively, as well as solutions by approximation and by "false hypothesis" (*regula falsi*), and lastly indeterminate equations of the first and second degrees respectively, "Diophantine analysis" and the rest. In the appendices are included (i) chronological tables showing the gradual development of algebraical notation, (ii) examples from the original works of the actual working-out of solutions of equations, from the Babylonians downwards to Newton and Leibniz—there are 76 of these examples, and there are also solutions of the equations $x^3 = a$ and $x^3 + bx = a$ by Omar Khayyam.

An outstanding feature of the volume is the space given to the new discoveries about Babylonian mathematics due to the decipherment and interpretation of cuneiform texts by O. Neugebauer, Kurt Vogel and others. It is now known that from 2000 and 1800 B.C. the Babylonians regularly worked with a sexagesimal system of numbers (including successive sexagesimal fractions) and solved quadratic equations (without ever stating any formulae) by the precise steps represented by the regular formulae of our textbooks. The publication of the original material by O. Neugebauer in many papers, in his history of *Vorgriechische Mathematik*, and finally in two large volumes of *Mathematische Keilschrifttexte*, with text, translation, commentaries and facsimiles, has given Herr Tropfke matter for many pages in this volume (see especially pp. 52-58). He has also taken pains to illustrate very elaborately (with many diagrams) the dependence of the first solutions of quadratic equations, by the Greeks and Arabs in particular, upon propositions in Euclid (Book II, especially Props. 4, 5, 6, and Book VI, 27-29, of the *Elements*, and Props. 84, 85 and 86 of the *Data*).

Throughout the book it is fascinating to notice the infinite variety of notations used in algebraical work from the original pioneers downwards, *e.g.* by Leonardo of Pisa, Jordanus Nemorarius, Luca Paciuoli, Christoph Rudolff, Stifel, Cardano, Bombelli, Stevin, Viète, van Schooten, Kepler, Harriot, Oughtred, Fermat, Descartes and the rest. If any one wishes to see how complicated the solution of the quadratic equation $x^2 + px = q$ can be made to look, and how difficult to follow, by the multiplication of letters a, b, c, etc., let him look at the solution by Jordanus Nemorarius on p. 83.

It is a pleasure to see included the solution by Archimedes of the equivalent of the cubic equation involved in his method of inscribing a regular heptagon in a circle (pp. 127-8).

The book is and will be an indispensable companion for all who would keep themselves up to date in the history of mathematics. T. L. H.

An Illustrated Historical Time Chart of Elementary Mathematics for Senior and Secondary Schools, Training Colleges and Universities. By E. J. Edwards. Five charts, thick cardboard, varnished. 21s. 1936. (University of London Press)

Nothing like this time chart has appeared before, so far as the reviewer knows, and there are few principles by which to judge it. Before mentioning

the interesting points which it raises, a general account of its purpose and a detailed description of its contents are necessary.

The purpose of the chart is to show the history of mathematics to the end of the seventeenth century, or a little beyond, by arranging in order the main stages, discoveries and pioneers in the subject. The chart is intended to be hung in the mathematical classroom. It is in five sections, each about $2' \times 3'$, of thick varnished cardboard. (Unfortunately, few schools have a mathematical classroom: the subject is usually pursued in many rooms, and perhaps schools which purchase the chart will find its divisibility into sections a convenient way of encouraging the teaching of the history of mathematics on a wide front.)

Each section has a general heading: "Beginnings of Mathematics", "Mathematics in Egypt, Babylonia and China", and so on. Beneath the heading is a horizontal date line and below this are ranged the principal facts of the history of mathematics during the period considered. The third section on "Mathematics in Ancient Greece" seems a particularly good piece of work. It contains summaries of the achievements of seventeen mathematicians. Those teachers who are prepared to name the seventeen principal Greek mathematicians, and to describe their principal discoveries, will have nothing to learn. Most teachers, however, will be interested in the information given, and they will be pleased by the instructive illustrations which enliven it. This section of the chart illustrates its purpose and achievement at their best. The others seem less good, for both their information and its arrangement may be criticised.

To take the arrangement first. Instead of printing, the chart has been executed in script writing of various sizes. This has enabled diagrams and symbols to be interposed freely, but it has made much of the chart difficult to read, because there is so little uniformity. As the sections are arranged on the classroom wall the headings, even, are not alike in size or level. Some of the minor facts are written in script less than a tenth of an inch in height, which means that the sections concerned must be hung at eye-level. This will bring them within reach of eager and possibly interfering fingers, and may result in damage to their surfaces. Fortunately the sections are washable. The style of the script writing is elaborate: considerable labour and skill have been used, but we feel that a less generous use of heads and tails to letters would have made the chart easier to follow.

The information given on the chart deserves serious consideration, because it makes one ask what should be taught in the history of mathematics. Looking at these sections one is confronted by a great collection of isolated facts, linked by a few generalisations. The facts may be criticised on the ground that some of them are of little value for school purposes, and the generalisations are sometimes vague and questionable. For instance, the first item on the chart is a rare collection of terms of indefinable vagueness: "It is probable that at a very early stage in the history of the race the primitive savage knew the difference between a large and a small flock." Nor does it seem correct to attach to the diagram showing the Great Pyramid of Gizeh the comment that "At this stage mathematics changes from mere counting to practical geometry." One weakness of the whole chart is the disproportion of information between antiquarian stuff about early mathematics and significant information about important discoveries. For instance, the final section on "Mathematics since 1500" contains notes on Bhaskara's Rule for Solving a Quadratic (the answer is incorrect), Finger Reckoning, Introduction of Calculating Machines, Maclaurin's Theorem, Amsler's Planimeter. This is only a selection, but not an unfair one. We are told that "Adelhard of Bath and Robert of

Chester translated Al-Khowarismi's Arithmetic into Latin "—hardly necessary for school purposes—and Kepler's life work is summed up by the sentence "stated his three laws of planetary motion ". Jordanus receives as much space as Kepler. These crude facts surely require a substantial expansion by the teacher if any good is to come from teaching the history of mathematics. For Kepler, the information in Lodge's *Pioneers of Science* may be useful. It is necessary for pupils to realise that Kepler had no analytical geometry, calculus or logarithms, that his discoveries resulted from fifteen years of trial and error, that the starting point was a trivial discrepancy in the calculated position of a planet, and that the whole research was steeped in astrology and mysticism. In the generalisations we should have liked to see emphasis on the enormous importance of a workable and creative symbolism in mathematics, on the fact that given such a symbolism the subject grows almost of its own accord, and the fact that the most useful symbol in mathematics is 0.

However, this sort of thing can be supplied by the teacher. The facts can be usefully displayed on classroom walls, and to show them there is certainly one way of introducing the history of mathematics. Teachers who adopt this time chart will take a step in the right direction—though they may find some awkward " snags " : " Please, Sir, what does the chart mean by saying that the principles used in Babylonian notation are additive and multiplicative ? What is alligation ? How does an abacus work ? What was Aristotle's dynamical proof of the Parallelogram of Forces ? "

The reviewer has hung his chart on the wall of his room : it has certainly aroused interest, and it has stimulated him and his pupils to hang up other exhibits of mathematical diagrams and models. The chart seems imperishable, and very good value for a guinea. We commend it to all teachers, and we hope that the authors, or others, will extend the idea. Too little use is made of visual aid in teaching mathematics. The poster and the film have still to be used freely. Indeed, the possibilities seem fascinating, and the material available enormous. How much would one not give for a few wall charts summarising those massive tomes on the history of the theory of numbers, or on the history of determinants ! C. T. Daltry.

The Collected Works of George Abram Miller. Volume I. Pp. xi, 475. \$7.50. 1935. (University of Illinois Press)

The name of Professor G. A. Miller is well known to all who interest themselves in the theory of groups of finite order. Professor Miller's original contributions to that subject now number approximately 400; and this total is fortunately still growing. His work has appeared over a period of more than forty years in a wide variety of mathematical periodicals. And it is a matter for congratulation that these papers are now being republished in collected form under the auspices of the University of Illinois Press.

The present volume covers the last five years of the nineteenth century and contains some sixty papers. It has been prepared mainly by Professor Miller himself, and he has supplemented it by three substantial and very valuable notes on the history of the theory prior to 1900. These notes were specially written with a view to placing his work in its proper historical perspective.

The decade 1890-9 may well be regarded as a turning point in the development of the theory of groups of finite order. Up to that time it had been predominantly a theory of permutation groups. (Congruence groups and the groups of the regular solids form an exception more apparent than real.) And the main influence had been the French school of Galois, Cauchy, Serret, Mathieu and Jordan. Towards the close of the century, however, a profound transformation took place. From being an appendix to the

theory of equations, groups began to be studied mainly for their own sake. The idea that a group necessarily consists of permutations or operations of some kind or other was gradually eliminated; and the notion of an " abstract " group, satisfying certain formal axioms, was evolved to take its place. And by contrast, permutation groups and groups of linear substitutions came to be regarded more and more as " concrete " representations of abstract groups.

With representation theory in the strict sense, as developed by Dyck, Frobenius, Burnside and Schur, Professor Miller is not concerned. He begins research, as a disciple of F. N. Cole, with pure permutation theory. And as the volume proceeds, we see his interests gradually moving in the direction of abstract groups, even while his methods still remain preponderantly permutational in character.

The most important papers included here are those which deal with the following two problems: the determination of all the permutation groups of given degree, or of all the transitive ones, or of all the primitive ones; and the determination of all the abstract groups of given order. In both of these problems, Professor Miller has achieved notable success. In the first of them, the work of the earlier writers had been somewhat experimental in method, and often inaccurate or incomplete in result. And we owe to Professor Miller, in particular, the first completely successful enumeration of the permutation groups of the degrees 8, 9 and 10.

With regard to the second of the two problems mentioned, much valuable work was done in the 'nineties, especially by J. W. A. Young, Hölder and Bagnera. And the volume under review also makes several most important contributions to its solution. To mention only one, there appear here for the first time (in the guise of regular permutation groups) the 15 groups of order 24 and the 51 groups of order 32; these are the two most difficult of the smaller orders.

The interest of these papers, however, is by no means confined to these enumerational questions. We find here the first general discussion of the commutator sub-group of a given group, for instance; and there are besides many illuminating extensions of classical theorems such as Sylow's, or Jordan's on uniprimitive groups, or Dedekind's on Hamiltonian groups; and much else that it is impossible to mention.

The printing, binding and indexing reach a very high standard of excellence, and we shall look forward eagerly to the appearance of the second volume. P. Hall.

Einführung in die analytische Geometrie und Algebra. II. By O. Schreier and E. Sperner. Pp. 308. Geh. RM. 6; Geb. RM. 7.20. 1935. Hamburger Mathematische Einzelschriften, 19. (Teubner)

The first volume of this work was reviewed in the *Gazette* for February, 1933. This final volume contains an introduction to the theory of (finite) matrices, which has already been published separately and is now out of print, together with a treatment of n-dimensional analytic geometry from the projective, affine and metric points of view, as far as the classification of quadrics.

The work on matrices is very thorough and detailed and uses methods of proof that have been developed comparatively recently, as, for example, in Weyl's *Raumproblem*. A comparison with an older book, Muth's *Elementartheiler*, will show the great increase achieved in simplification and clarity. It is customary in most presentations to begin with the reduction of matrices of integers; in this book this is disguised or utilised as a proof of the fundamental theorem on the basis of Abelian groups; in this way the proof becomes natural, and loses the artificial *ad hoc* appearance that it has in books on the

theory of groups or of numbers. Moreover, the investigation here applies to infinite groups, if they have a finite basis.

It is pleasant to find that matrices are connected always with linear substitutions and are given a geometric complexion; this is particularly helpful in connection with matrices of polynomials and elementary divisors. Special sections deal with the first properties of unitary, orthogonal and Hermitian matrices. The singular case is not considered.

The projective and affine geometry is developed analytically; thus the projective constructions for sums and products, which are used to set up coordinates in synthetic geometry, here enter as illustrations and not as building stones. Great care is taken in the classification of quadrics from the projective, affine and metric standpoints, and I do not know any other place where this is done so thoroughly. Only the general theory is given, no special properties are mentioned but, as all details are set out, no strain is imposed on the reader.

The book must be regarded as introductory both in scope and method, for it is concerned only with essentials, and does not use the more abstract means we have learned from Emmy Noether and van der Waerden. Thus though a careful distinction is drawn between real and imaginary fields when quadrics are classified, the importance of the fundamental theorem of algebra in connection with canonical forms of matrices is not stressed. We have essentially a textbook, and a student who finds van der Waerden or Weyl too concise and abstract would benefit, if he has sufficient staying power, by taking a preliminary course here; and though one misses the thrill and sense of adventure that is imparted in reading the masterpieces, this result of conscientious good workmanship should be consulted by those lecturing on the subject.

H. G. F.

Dynamics. Part II. By A. S. Ramsey. Pp. xi, 344. 15s. 1937. (Cambridge University Press)

This is a very good book and a welcome addition to Mr. Ramsey's series of textbooks.

The *Treatise on Dynamics* of Besant and Ramsey, which was published in 1914, was the last (fifth) edition of the late Dr. W. H. Besant's treatise, revised and expanded by Mr. Ramsey. It was a very useful book; it was concise and clear, and met the admitted need in dynamics for numerous examples by providing over a thousand for solution. It has been out of print for some years, to the regret of many teachers of mechanics, who will be glad that Mr. Ramsey has written the present book, which, though covering a wider field, is on much the same lines as the earlier one, to take its place.

The book continues the same author's recent *Dynamics*, Part I, which was written for first-year students at Universities, with an account of the three-dimensional motion of a particle and of a rigid body, and with a very clear introduction to the general theory of dynamics. It presents a very good course for second-year students, and for most third-year students, at Universities, and it is full, following the excellent practice of Mr. Ramsey's series, of good examples.

It should perhaps be said that no use is made in the text of vectors, and it may be hoped that Mr. Ramsey has opened an interesting discussion by expressing doubt whether in mechanics " it (*i.e.* the use of vectors) is a fashion which has come to stay ". An Appendix, however, gives an adequate account of the application of vector analysis to three-dimensional dynamics.

W. R. D.

Statistical Methods in Biology, Medicine and Psychology. By C. B. Davenport and Merle P. Ekas. Fourth Edition. Pp. xii, 216. 13s. 6d. 1936. (Chapman & Hall)

This book is one of the classics in the science of biometry, in the sense that it was first published in 1899, within a very few years of the first important memoirs on the subject published by Karl Pearson. It says much for the industry and perseverance of the authors that in this completely revised edition the book should be so up to date. Tests of significance of differences between means by the t-distribution, between variance estimates by the z-distribution, and the analysis of variance, are all dealt with, although the treatment of the last named is far from being commensurate with the important place this analysis now occupies in the literature of the subject, especially in its applications to biology. It is refreshing to see the exact test for significance of a correlation coefficient put first, and the familiar $(1-r^2)/\sqrt{n}$ second, with the conditions under which it is applicable carefully stated.

Reviewed as a whole the book may perhaps be best expressed by saying that it is statistics in tabloid form. The book proper covers 139 pages only, and within that compass are comprised chapters on variation and its measurement, on the seriation and plotting of data and the frequency polygon, on the classes of frequency polygon, covering most of Pearson's types, on analysis of variance, on correlated variability and measures of relationship, on heredity, and a final chapter on special topics, covering growth laws, index numbers and measurements of secular, seasonal and cyclical change. Numerous topics are dealt with, which makes the book valuable for reference, but little space is devoted to each, and in many cases a problem is stated, the solution given (without proof), and an illustrative example provided, all in a very short space. The latter half of the book is taken up with a list of references, an explanation of the tables, list of definitions and formulae, and a considerable number of tables designed to facilitate computation and to make the book self-contained.

One point of criticism concerns the stress laid on the frequency polygon as a description of discontinuous data, and the meagre references to continuous data, at least in so far as the fitting of frequency curves is concerned. Thus Pearson's types, which are continuous curves, are used to fit discrete data, and the calculated *ordinates* are called theoretical *frequencies*, corresponding to the observed frequencies. The grouping correction due to Sheppard is given on p. 43 as 1/12 (for the second moment) without explanation, and on p. 99 as $i/12$ instead of $i^2/12$, where i is the size of the class interval. The Index is full, but not full enough, for a sample test revealed no reference to "correction for grouping", "grouping" or "Sheppard". The concept "degrees of freedom" is discussed, but it is scarcely adequate to say on p. 48 in connection with goodness of fit that "this is at least one less than the actual number of classes". The student will want to know exactly how to enter the table of χ^2, but he will be misled by the statement on p. 59 that "degrees of freedom $=10-3$". In this case of fitting Type I five degrees of freedom have been used up, not three.

The book is interesting and well informed throughout, and should be very helpful as a laboratory manual, and in stimulating the further reading of the student of statistics. J. W.

Interpolation and Allied Tables. Reprinted from the *Nautical Almanac* for 1937. Pp. 46. 1s. 1936. (H.M. Stationery Office)

Published with a prefatory note by Dr. L. J. Comrie, at the time Super intendent of the *Nautical Almanac* Office, this booklet contains pp. 784-809 and pp. 926-941 of the *Nautical Almanac* for 1937, *i.e.* those portions which

give, with explanations, tables for facilitating interpolation. They should thus be of use generally in mathematical circles. In general it may be said that the object is to reduce interpolation and similar operations, such as numerical integration, to a straightforward computational basis. The tables given are: Table XIX—Besselian Interpolation Coefficients, giving the n (fraction of interval) for 0·001 ranges of the coefficients B^{II}, B^{III} and B^{IV}; Table XX—Lagrange Coefficients at 0·01 intervals of n; Table XXI—Everett Coefficients of the Second Difference, giving the n for 0·001 ranges of the coefficients; Table XXII—Throw-back from Fourth to Second Differences; Table XXIII—Second Difference Correction; Table XXIV—As in XIX, but for 0·0001 ranges of the coefficients; Table XXV—Besselian Interpolation Coefficients at 0·001 intervals of n; Table XXVI—Everett and other Interpolation Coefficients at 0·01 intervals of n; Table XXVIII—Coefficients for computing Derivatives from Differences. Tables XXVII and XXIX-XXXI are tables of formulae, for derivatives in terms of differences, differences in terms of derivatives, differences in sub-divided intervals and formulae for numerical integration.

The explanation is a very full one, and deals with the Bessel, Everett and Lagrange formulae of interpolation, a description of the tables, the use of calculating machines in this connection, graduated examples of direct and inverse interpolation, and finally a discussion of the calculation of derivatives and of sub-tabulation.

Unusual features are the inverse nature of some of the tables of interpolation, and the "throw-back". The first consists in tabulating the limiting values of the independent variable for which the dependent variable has a fixed value. A table such as this is used without interpolation and has a maximum error of 0·5 units in the last decimal. The "throw-back" is a device for reducing the number of even differences which must be tabulated with the function. For example, if Δ^{IV} is less than 1000, Δ^{II} may be replaced by the "modified" second difference $M^{II} = \Delta^{II} - 0{\cdot}184\Delta^{IV}$, with an error which does not exceed 0·46 units of the last decimal, and an interpolation formula going up to second (modified) differences only is used. Formulae of this kind up to eighth differences are given.

The booklet is a valuable addition to the literature on the subject of interpolation methods, and should be studied with care by all who have much of this kind of work to do. The *Nautical Almanac* Office has notified one correction. On p. 929, in the double "throw-back" formula for M^{IV}, the coefficient of Δ^{VI}, which is shown as 0·28727, should read 0·27827. J. W.

Pension and Widows' and Orphans' Funds. By D. A. Porteous. Pp. xii, 111. 7s. 6d. 1936. (Published for the Institute of Actuaries Students' Society at the University Press, Cambridge)

As this book is intended for students about to begin the subject, it naturally restricts itself to the mere preliminaries of the subject.

Owing to the great variety of the conditions which are sometimes introduced into pension funds, each one being a law unto itself, it is necessary for the actuary who ventures to value such funds to have ready to his hand a mass of material scattered over many publications and to have had experience under an actuary who has made a practice of such work. The mathematics necessary is in its broad outlines of the same nature as for ordinary life insurance valuations, with the introduction, however, of other factors, such as salary scale, rates of retirement (various for different businesses), rates of marriage and remarriage, and those of fertility. Information about these various rates are scattered over many journals. The author has been success-

ful in drawing up a well-ordered and clear preliminary summary of the subject. The student will do well to take it as a guide to further detailed work. References to a few of the more necessary papers are given. W. S.

Sur la Théorie Mathématique des Jeux de Hasard et de Réflexion. By René de Pôssel. Pp. ii, 44. 10 fr. 1936. Actualités scientifiques et industrielles, 436; conférences du Centre Universitaire méditerranéen de Nice, I. (Hermann, Paris)

In the first paragraph, headed " Généralités ", the author considers the factors which enter into the games of Society, and considers them of three kinds: la réflexion, le hasard and la ruse, which we may perhaps put down as choice, chance and chicanery, this last word being only a portion of the meaning of " ruse ". The author points out that these may enter, singly or jointly, as in chess or draughts, where we have choice only; dice, head or tail, etc., where we have chance only; and such a game as poker, which is mostly ruse or chicanery.

In paragraph two the games with " chips " (*batonnets*) is mentioned; thus if you have a heap of chips, and each player in turn takes one or two chips at his choice, the game is won by the player who takes the last chip. The result for each player is determined in advance by one who knows the number of chips in the heap. The author gives an analysis in the general case of n chips, and shows the use of the binary scale in such problems.

In paragraph three a fundamental theorem, where combinations enter and the players have some choice in their moves, is dealt with.

In the fourth paragraph the author considers games of chance, expectation, fair play, and the " ruin of players ", dealing especially with roulette, with the influence of a martingale on the results of play. Those who are interested in roulette may read with benefit a paper by Prof. Karl Pearson on " The Scientific Aspect of Monte Carlo Roulette ", given in Vol. I of *The Chances of Death, etc.*, pp. 42-62. In this paper Prof. Pearson sums up as follows: " Monte Carlo roulette, if judged by returns which are published without apparently being repudiated by the Société, is, if the laws of chance rule, from the standpoint of exact science the most prodigious miracle of the nineteenth century."

Perhaps here we come upon a good example of " chicanery ", which presents itself in its highest form in the game poker, where a false face is the greatest asset.

In the fifth and last paragraph the author touches upon the " General Theory of the Games of Society ", following mainly the work by von Neumann (*Mathematische Annalen*, Vol. 100, 1927) in his paper, " Über die Gesellschaftspiele " (" On the Games of Society ").

The study of the behaviour of the players, during a great number of games, may be a subject of interest and well worth studying, but any attempt to use it as a means of gain in gambling can hardly be recommended on moral grounds; and in the amusing *Familiar Colloquy* by Erasmus, which concerns gaming, we sympathise with Quirinus in his desire to play for nothing, when Carolus retorts, " Would you learn such an art for nothing ? " Quirinus replies: " It is truly an unequal match for a beginner to play with a gamester." *Veram impar certamen est inter artificem et rudem.* W. S.

L'algèbre abstraite. By Oystein Ore. Pp. 52. 15 fr. 1936. Actualités scientifiques et industrielles, 362; exposés d'analyse générale, VI. (Hermann)

Professor Oystein Ore's work is a brief survey of the more important theorems of the purely formal mathematics known as abstract algebra. It deals with the representation of number systems, with abstract groups and

corps, as well as with the more general investigations of those ranges of knowledge where a given theory is true and with those universal theorems relating to the structure of a mathematical system.

These matters are of special interest to those wishing to study their own field of mathematics from what is sometimes termed the higher standpoint, assisting as they do in the search for analogies and generalisations and in the coordinating of the various branches of the subject.

The titles of the successive paragraphs in the book are as follows:

Classification des systèmes algébriques, Les Corps, Corps topologiques, Les anneaux commutatifs et la théorie des idéaux, Théorèmes de décomposition pour les idéaux, Applications, Anneaux entièrement fermés, Anneaux non-commutatifs, Les systèmes hypercomplexes, Représentations des systèmes hypercomplexes, Les groupes, Structures. A. R. R.

The Analytical Geometry of Conic Sections. By B. B. Bagi. Pp. viii, 248. Rs. 4.12 as. 1936. (Published by the author, Gibb Town, Dharwar, India)

It is difficult to write a fair review of this book, because it is written for such a definite purpose, namely the needs of the students of the University of Bombay—so much the author makes clear in his preface. It is so unusual, for example, to find chapters on the equations of the ellipse, hyperbola and parabola, referred to axes of symmetry, following on, instead of preceding, general considerations. On the other hand, it is not unreasonable that the polar equation of the conic should be deferred to the final chapter in a book which does not include a treatment of homogeneous coordinates. It is no doubt logical to proceed from the general to the particular, but to do so is not in accordance with teaching practice in most other countries, and for this reason this book is not likely to find general acceptance.

Further, the reviewer thinks it wiser not to introduce oblique coordinates so early: it is more usual to recapitulate work in rectangular coordinates in connection with the new method of reference. The treatment of oblique axes, however, is very satisfactory, beginning, as it does, with

$$(x-x_0)/l=(y-y_0)/m\equiv r,$$

where

$$l^2+2lm\cos\omega+m^2=1.$$

In Ex. IIIa (p. 37) a number of examples are given on lines connected with a triangle, the coordinates of whose vertices are given. It is a pity that in Ex. (6) and (7) the internal and external bisectors of the angle A are respectively given as

$$AC\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} \pm AB \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix}=0.$$

These should be written

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} \div CA = \pm \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} \div AB,$$

so that the student may be able to write down at once the four sets of equations giving the coordinates of the incentre and excentres.

Very free use of determinants is made, and early in the book the third invariant, $\Delta/\sin^2\omega$, is proved to exist. This rather troublesome piece of work having been done, good use is made of the result later on.

The examples are for the most part a large and good selection from recent university examination papers, and answers are given at the end of the book.

The table of contents is very full, and it is quite easy to pick out any part of the subject matter without loss of time.

On the whole this book makes a very interesting addition to a mathematical library; the teacher will find it stimulating, but the student will find it difficult. N. M. G.

Numbers and Numerals. By David Eugene Smith and Jekuthial Ginsburg. Pp. 52. 25 cents. 1937. (Bureau of Publications, Teachers College, Columbia University, New York City)

Numbers and Numerals is the first of a series of monographs on *Contributions of Mathematics to Civilization*, edited by W. D. Reeve, and sponsored by *The Mathematics Teacher*, the official organ of the National Council of Teachers of Mathematics. It is suggested that these monographs could be used by teachers of either mathematics or the social studies in their classes as supplementary reading material.

The first five chapters of *Numbers and Numerals* are historical, and deal with counting scales, number names, numerals, computation with apparatus and with numerals and, very briefly, with fractions. The two following chapters are more composite. They contain curious obsolete customs and ideas about numbers and more modern number knowledge, introduced wherever possible as number games and puzzles. The last chapter gives the derivations of arithmetical terms. There is a map of the Old World showing the countries mentioned, an index, and the titles of Professor D. E. Smith's fuller works and other books for further reading.

The pupils who use this book will be fortunate. They will read what others have not read—because history of mathematics is so rarely included in a school curriculum—the story of one of the great factors in civilization. At successive stages the authors show a new need arising and describe man's efforts to increase mathematics and mould it to the purpose, until they have given all the essential facts about numbers and computation and the development, form and use of the numerals of all times. To cover the ground of whole numbers so completely in thirty-four pages is a masterpiece of condensation, more noticeable because the matter is given in simple words and explained as to a beginner. The illustrations are plentiful and fascinating, whether they range from the numerals and suan-pan of the Chinese to the quipu and counting-board of the Peruvians, or trace the development of the Hindu Arabic numerals from their origin to the printed figures of to-day. The reader is given the opportunity of trying his skill in early modes of reckoning.

The book is interesting from cover to cover, and can be read by pupils of thirteen years and older. F. A. Y.

Graphs and Statistics. By John Maclean. Pp. xiii, 200. Rs. 4. 1926. (Obtainable from Ramchandra Govind & Son, Bombay 2)

This book is written primarily for the Indian student at a time when Mathematics was a compulsory subject in the First Year.

The author's sub-title is " A suggestion for a finishing course in Mathematics ". In his preface he says that " All that has been attempted is to put the student in a position to appreciate some of the main possibilities and limitations of statistical methods ". Later to the student he writes: " In this book you see Mathematics as a humble servant of the other sciences and you get only glimpses of her queenly glory [and see] something of her methods of orderliness and elasticity, consistency and thoroughness, care and brevity of representation."

The book assumes a knowledge of logarithms, similar triangles, an introduction to probability. Chapter I is " Miscellanea ", touching parallel displacement of axes, solution of cubics, accuracy of reading graphs, discovery of laws, graphical and algebraical approximations, binomial expansions and determinants, all in 21 pages. Chapter II deals with " typical " graphs—parabolic, hyperbolic, cubic on various axes, logarithmic and exponential curves, and later the periodic curves. Chapter III, *via* approximations, we find ourselves in the infinitesimal calculus, the slopes of curves, maxima and minima and area under the curve.

Chapter IV is a very good chapter on the slide rule, emphasising that it is a means of adding and subtracting indices, and that it can be adapted for many purposes. Chapter V deals with that aspect or practical use of scales, which is little treated in schools, the nomogram. The explanations are very lucid and the subject becomes distinctly fascinating, especially on p. 72 when one finds oneself reading the capacity of a tank in cubic centimetres, cubic feet, gallons and tons or pounds of water just as desired. Here, as often elsewhere, the subject matter becomes bewildering, and some of the composite nomograms, one feels, would leave the non-specialist very cold.

Chapter VI, entitled " Typical Numbers ", brings us those familiar terms, Median, Deviation, Quartiles, Standard deviation, Correlation coefficient, and this is carried on in Chapter VII to Frequency tables, histograms, while the measurements of correlation coefficients are further discussed, and Chapter VIII deals with Probability and the prediction of events. Throughout, biological factors form the basis of many examples, tables and curves, and Chapter IX, the concluding chapter of 60 pages, is devoted to the use of logarithmic rulings, trilinear coordinates and periodic rulings in dealing with results from the biological sciences.

This review may appear to be a succession of chapter " contents " but each chapter is in itself a comprehensive whole rendering a general survey inadequate.

E. J. A.

Descriptive Mathematics. By John Maclean. Pp. xvi, 135. Rs. 2.8. 1935. (Macmillan, Bombay and London)

" The origin of this book is twofold. Its contents are largely the results of a search . . . through recent scientific writings for uses of elementary mathematical methods in the description of quantitative phenomena. . . . In contrast with *Graphs and Statistics* (1926), where the stress was laid so much on the applications that mathematicians found but little interest in it, the emphasis here is often intensely mathematical."

The writer has in mind the idea of creating in the student the desire to do things for himself. In the opening chapter he urges the making of a slide rule by each pupil, a point which has been advocated in the *Gazette.* Chapter II deals with Cartesian coordinates, but a clear introduction is illustrated by graphs which are bewildering by their complexity, and in a further paragraph or so we are led through the consideration of the parabola, cubic curves, logarithmic and exponential curves, turning points, points of inflexion and areas. Chapter III takes us through the spiral, circular functions, and irrational numbers. Chapters IV-VII deal respectively with nomograms, statistics, probability and finite differences. All this is included in 135 pages.

The author tells us that he is developing an experiment in attempting to interest the non-mathematical specialist. The amount of ground covered in the very small space is enormous. It is difficult to imagine the ordinary non-mathematical boy in this country picking up this book and feeling that he had discovered a long-lost treasure. The rate of progress is such that the com-

parison of a non-swimmer thrown into the middle of a deep expansive lake to the boy with this book would not be unreasonable.

There are truly many fascinating sections of mathematical study opened to the view, but there seems to be a lack of that progressive sequence which is so very necessary. The illustrations, too, are frequently involved matters of specialised material.

One admires the author's courage but one does feel that there must be much supplementing of the text, for in this later book in 135 pages the author bridges the gap between the solution of a pair of simple equations and the method of finite differences. Further, if the non-mathematical student has enjoyed and appreciated all that Professor Maclean has put before him in this text, I submit that he will cease to be a non-mathematical student.

E. J. A.

Elemente der Funktionentheorie. By K. Knopp. Pp. 144. RM. 1.62. 1937. Sammlung Göschen, 1109. (Walter de Gruyter)

Funktionentheorie. I. Grundlagen der allgemeinen Theorie der analytischen Funktionen. By K. Knopp. 5th edition. Pp. 136. RM. 1.62. 1937. Sammlung Göschen, 668. (Walter de Gruyter)

Knopp's two small Sammlung Göschen volumes on *Funktionentheorie* (the complex variable) form a clear, accurate account of the groundwork of the subject. Volume I, now appearing in a fifth edition, deals, after a rapid discussion of preliminaries, with the classical material on integration, expansions and singularities. Of necessity, there is little room for illustrations or applications, but the theory is developed admirably.

The *Elemente der Funktionentheorie* is a new volume serving as an introduction to the *Funktionentheorie*; here the brief account of preliminary matter given in Volume I of the *Funktionentheorie* is expanded and illustrated. In the first section, the axiomatic treatment of complex numbers is given, with strong emphasis on geometrical illustration. Then there are sections on the "linear" function $w=(az+b)/(cz+d)$ and the geometrical transformations to which it gives rise; on sets, sequences and power-series; on the concept of an analytic function, and conformal representation; and on the elementary functions of a complex variable.

Much novelty of treatment is not to be expected in so classical a domain; but the teacher of this subject who finds himself faced with problems not only of presentation but of selection will be well advised to study these excellent little volumes. T. A. A. B.

BUREAU FOR THE SOLUTION OF PROBLEMS.

This is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. The names of those sending the questions will not be published.

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